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On the multipacking number of grid graphs

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Abstract

In 2001, Erwin introduced *broadcast domination* in graphs. It is a variant of classical domination where selected vertices may have different domination powers. The minimum cost of a dominating broadcast in a graph G is denoted $\gamma_b(G)$. The dual of this problem is called *multipacking*: a multipacking is a set $M \subseteq V(G)$ such that for any vertex v and any positive integer r , the ball of radius r around v contains at most r vertices of M . The maximum size of a multipacking in a graph G is denoted $\text{mp}(G)$. Naturally $\text{mp}(G) \leq \gamma_b(G)$. Earlier results by Farber and by Lubiw show that broadcast and multipacking numbers are equal for strongly chordal graphs.

In this paper, we show that all large grids (height at least 4 and width at least 7), which are far from being chordal, have their broadcast and multipacking numbers equal.

Introduction

Given a graph G with vertex set V and edge set E , a *dominating broadcast* of G is a function f from V to $\mathbb{N}$ such that for any vertex u in V , there is a vertex v in V with $f(v)$ positive and greater than the distance from u to v . Define the *ball of radius r around v* by $N_r(v) = \{u : d(u, v) \leq r\}$. Thus a dominating broadcast is a cover of the graph with balls of several positive radii. The *cost* of a dominating broadcast f is $\sum_{v \in V} f(v)$ and the minimum cost of a dominating broadcast in G , its *broadcast number*, is denoted $\gamma_b(G)$.

Remark. One may consider the cost to be any function of the powers (for example the sum of the squares), see e.g. [10]. We shall stick to the classical convention of linear cost.

The dual problem of broadcast domination is *multipacking*. A multipacking in a graph G is a subset M of its vertices such that for any positive integer r and any vertex v in V , the ball of radius r centred at v contains at most r vertices of M . The maximum size of a multipacking of G , its *multipacking number*, is denoted $\text{mp}(G)$. We may write γ_b and mp when the graph in question is clear from context or unimportant.

Broadcast domination was introduced by Erwin [7, 8] in his doctoral thesis in 2001. Multipacking was then defined in Teshima's Master's Thesis [13] in

2012, see also [1]. However, this work fits into the general study of coverings and packings, which has a rich history in Graph Theory, see for example the monograph by Cornuéjols [3].

Since minimum dominating broadcast and multipacking are dual problems, we know that for any graph G ,

$$\text{mp}(G) \leq \gamma_b(G).$$

A natural question comes to mind. Under which conditions are they equal? For example, it is known that strongly chordal graphs have their broadcast and multipacking numbers equal. This follows from a primal-dual algorithm of Farber [9] applied to Γ -free matrices, used to solve the (weighted) dominating set problem for strongly chordal graphs. The work of Lubiw [11, 12] shows the vertex-neighbourhood ball incidence matrix is Γ -free for strongly chordal graphs, and hence the primal-dual algorithm can also be used to solve the broadcast domination problem for strongly chordal graphs. For trees, direct proofs of $\text{mp}(T) = \gamma_b(T)$ and linear-time algorithms to find $\text{mp}(T)$ appear in [1, 2] (see also [4, 5]). For strongly chordal graphs, Farber's algorithm runs in $O(n^3)$ time. The general broadcast domination problem can be solved in $O(n^6)$ time [10]. In this paper we study grid graphs which are far from being strongly chordal (or even chordal). We show the following theorem.

Theorem 1. *For any pair of integers $n \geq 4$ and $m \geq 4$,*

$$\text{mp}(P_n \square P_m) = \gamma_b(P_n \square P_m).$$

with the exception of $P_4 \square P_6$ where $\text{mp}(P_4 \square P_6) = 4$ and $\gamma_b(P_4 \square P_6) = 5$.

This gives an infinite family of non-chordal graphs for which $\text{mp} = \gamma_b$. Another such family is the cycles of length 0 modulo 3. It is trivial to verify that $\text{mp}(C_{3k}) = \gamma_b(C_{3k}) = k$.

Dunbar et al. [6] gave the exact value of the broadcast number for grids.

Theorem 2 (Dunbar et al. [6, Th. 28]). *For any pair of positive integers n and m ,*

$$\gamma_b(P_n \square P_m) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor.$$

Remark. The value of $\gamma_b(P_n \square P_m)$ given by Theorem 2 is the radius of the grid. Since there is always a dominating broadcast with cost $\text{rad}(G)$ [6, 8], and our proof of Theorem 1 yields a multipacking of size $\text{rad}(G)$, this paper gives an alternative proof of Theorem 2.

1 Preliminaries and small grids

We use standard notation throughout the paper. Specific to our work is the following: the grid $P_n \square P_m$ has n rows and m columns. We may also say the

grid has height n and length m . The vertex in row i and column j is denoted $v_{i,j}$. As a convention, the vertex $v_{0,0}$ is the bottom, left corner of the grid. The integers from between k and l inclusive are denoted $\llbracket k, l \rrbracket$.

The proof of Theorem 1 is technical. In order to ease the process, we start with an easy counting lemma.

Lemma 3. *Let G be a graph, k be a positive integer and $u_0, \dots, u_{3k}$ be an isometric path in G . Let $P = \{u_{3i} : i \in \llbracket 0, k \rrbracket\}$ be the set of every third vertex on this path. Then, for any positive integer r and any ball B of radius r in G ,*

$$|B \cap P| \leq \left\lceil \frac{2r+1}{3} \right\rceil.$$

Proof. Let B be a ball of radius r in G , then any two vertices in B are at distance at most $2r$. Since the path $(u_0, \dots, u_{3k})$ is isometric the intersection of the path and B is included in a subpath of length $2r$. This subpath contains at most $2r+1$ vertices and only one third of those vertices can be in P . $\square$

For the sake of completeness, we also determine the multipacking numbers of grids with height 2 and 3.

Proposition 4. *Let n be a positive integer. Then*

$$\text{mp}(P_n \square P_2) = \left\lceil \frac{2n}{5} \right\rceil$$

Proof. Let P be a maximum multipacking of $P_n \square P_2$. We claim that no five consecutive columns contain three members of P . Suppose to the contrary that columns i to $i+4$ contain three members of P . No two consecutive columns each contain a member of P , as any pair of vertices in $P_2 \square P_2$ are at distance at most 2 apart (and thus in a ball of radius 1). Hence, the three elements are without loss of generality $\{v_{i,0}, v_{i+2,1}, v_{i+4,0}\}$. However, this implies $|N_2[v_{i+2,0}] \cap P| = 3$, a contradiction.

Writing $n = 5q + r, 0 \leq r \leq 4$, we conclude that the first $5q$ columns of the grid contain at most $2q$ elements of P . Next, it is easy to verify that $\text{mp}(P_1 \square P_2) = \text{mp}(P_2 \square P_2) = 1$, and $\text{mp}(P_3 \square P_2) = \text{mp}(P_4 \square P_2) = 2$. Let s be the number of elements of P in the final r columns of the grid. Then, $s = 0$ if $r = 0$, $s \leq 1$ if $r = 1, 2$ and $s \leq 2$ if $r = 3, 4$. Thus, $|P| \leq 2q + \lceil 2r/5 \rceil$. Equivalently, $|P| \leq \lceil \frac{2n}{5} \rceil$.

On the other hand, consider the set P defined as follows.

$$\begin{aligned} v_{i,0} &\in M & \text{for } i &\equiv 0 \pmod{5} \\ v_{i,1} &\in M & \text{for } i &\equiv 2 \pmod{5} \end{aligned}$$

Consider a ball B of radius $r \geq 2$. It contains vertices from at most $2r+1$ consecutive columns of $P_n \square P_2$. By construction, every five consecutive columns contain at most 2 elements of M .

$$|B \cap P| \leq 2 \left\lceil \frac{(2r+1)}{5} \right\rceil$$

It is straightforward to check that, $2\lceil(2r+1)/5\rceil \leq r$ for $r \neq 1, 3, 5$. (For $r < 10$, simply evaluate $2\lceil(2r+1)/5\rceil$. For $r \geq 10$, $2\lceil(2r+1)/5\rceil \leq 2(2r/5+1) \leq r$.) It is easy to check that each ball of radius 1 contains at most one element of P . When $r = 3$, B contains vertices from 7 columns of which at most 3 columns may contain packing vertices. Similarly, when $r = 5$, we observe that any 11 consecutive columns contain at most 5 packing vertices. $\square$

We now turn to the special case when $m = 3$. The following result gives $\text{mp}(P_n \square P_3)$. Since $\gamma_b(P_n \square P_3) = \lfloor \frac{n}{2} \rfloor + 1$, we note that $\text{mp} = \gamma_b$ for $n \not\equiv 0 \pmod{4}$.

Proposition 5. *Let n be a positive integer. Then*

$$\text{mp}(P_n \square P_3) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } n \equiv 0 \pmod{4} \\ \lfloor \frac{n}{2} \rfloor + 1 & \text{if } n \equiv 1, 2, 3 \pmod{4} \end{cases}$$

Proof. Since $\text{rad}(P_n \square P_3) = \lfloor n/2 \rfloor + \lfloor 3/2 \rfloor$, we know that $\text{mp}(P_n \square P_3) \leq \lfloor n/2 \rfloor + 1$. Given a multipacking P of $P_n \square P_3$ and any four consecutive columns say $i, i+1, i+2, i+3$, if P has three members in these columns, then without loss of generality they belong to columns $i, i+2, i+3$. Moreover, we can assume that the two in columns $i+2, i+3$ are $v_{i+2,2}$ and $v_{i+3,0}$. The only vertices that are not within distance 2 of either of these two packing vertices are $v_{i,0}$ and $v_{i,1}$. However, all three of these vertices are in a ball of radius 2 centred at $v_{i+2,0}$ in the former case and $v_{i+2,1}$ in the latter, a contradiction. Thus, the four columns contain at most 2 packing vertices. Specifically, in the case $n = 4q$, $\text{mp}(P_n \square P_3) \leq 2q = \lfloor n/2 \rfloor$.

On the other hand, consider the set P defined as follows.

$$\begin{aligned} v_{i,0} &\in M & \text{for } i &\equiv 0 \pmod{4} \\ v_{i,2} &\in M & \text{for } i &\equiv 1 \pmod{4} \end{aligned}$$

As the minimum distance between vertices in P is 3, no ball of radius 1 contains more than one element of P . Consider a ball B of radius $r \geq 2$. The ball contains vertices from at most $2r+1$ consecutive columns. We need to confirm that the ball has at most r elements of P . First, suppose that $r = 2t$. By symmetry, we may assume that the left most column of B is in $\{0, 1, 2, 3\}$. If the left most column is 3 or 4, then B contains vertices from columns $\{4, 5, \dots, 4t+2, 4t+3\}$. Each contiguous block of four columns contains two members of P , giving B has a total of at most $2t = r$ vertices of P . If the left most column of B is 0 or 1, then B covers columns $0, 1, \dots, 4t$ or $1, \dots, 4t, 4t+1$. In both cases, B has exactly $2t+1$ columns with a vertex of P . However, in both cases $v_{1,2}$ and $v_{4t,0}$ are at distance $4t+1 = 2r+1$ apart and thus, at most one belongs to B . In all cases, $|B \cap P| \leq r$. If $r = 2t+1$, the analysis is similar. Either the $4t+3$ columns of B contain at most $2t+1 = r$ vertices of P , or the ball B has $2t+2 = r+1$ columns containing vertices of P , but there is a pair (for example $\{v_{0,0}, v_{4t+1,2}\}$) at distance $2r+1$, in which case B itself contains at most r vertices of P . $\square$

2 Multipacking number for large grids

In this section, we prove Theorem 1. The radius of a grid graph $P_n \square P_m$ is $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$. Since the broadcast number of a graph is at most its radius, it is sufficient to find a multipacking of size $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$. We now proceed with the construction of such multipackings.

2.1 Restriction to even sizes

First, we shall prove that we can restrict ourselves to cases when n and m are both even numbers. Because of the singularity for the grid of size 4×6 , we need to check the grids of sizes 5×6 and 4×7 by hand (see Figure 1). Now, suppose that n is odd. Then, $n - 1$ is even and is at least 4. Moreover the $n - 1 \times m$ grid is not 4×6 since we ruled out the 5×6 and 4×7 cases. Thus, if we know that the grid of size $n - 1 \times m$ has a multipacking of size $\frac{n-1}{2} + \lfloor \frac{m}{2} \rfloor$, which is equal to $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$, we can add an empty column in the middle of this grid. We obtain a multipacking of the desired size for our grid. Indeed, given a vertex v from the smaller grid, the ball of radius r with centre v in the larger grid only contains vertices of the packing which were at distance at most r from v in the smaller grid. A ball of radius r centred at a vertex of the new column only contains vertices of the packing which are within distance r of both its neighbours from the former grid. Thus, these balls cannot contain more than r elements of the packing which satisfies our claim. The same reasoning works for m .

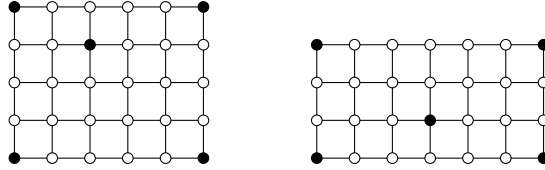


Figure 1: Multipackings of order 5 for grids of size 5×6 and 4×7

The remainder of the proof is concerned with grids with even dimensions. Small cases require some specific care so that we will treat them after the general case. In all cases, we shall use a systematic way of selecting vertices along the sides of the grid. We describe them in the following paragraph.

2.2 The i -pattern

Fix an integer i . Given a path $v_0 v_1 \dots v_{z-1}$ of order z greater than or equal to $3i$, the i -pattern on this path consists in selecting every third vertex from v_0 to $v_{3(i-1)}$ and then every fourth vertex starting from v_{3i} (if it exists). Note that the i -pattern on a path of order z selects exactly i vertices from the beginning and one fourth (rounded up) of the rest. This amounts to $i + \lceil \frac{z-3i}{4} \rceil$ which can

be simplified.

$$\text{The } i\text{-pattern on a path of order } z \text{ selects exactly } \left\lceil \frac{z+i}{4} \right\rceil \text{ vertices.} \quad (1)$$

Moreover, the density of the i -pattern is bounded above by a function of i . By this, we mean that a subpath of length ℓ of $v_0v_1 \dots v_{z-1}$ cannot hit too many vertices of the i -pattern. If ℓ is at least $3i$, it could take the whole beginning (i vertices) and a fourth of the rest. This amounts to $i + \lceil \frac{\ell+1-3i}{4} \rceil$ which equals $\lceil \frac{\ell+1+i}{4} \rceil$. Whenever ℓ is strictly less than $3i$, it would take at most $\lceil \frac{\ell+1}{3} \rceil$ vertices. But in that case,

$$\begin{aligned} \left\lceil \frac{\ell+1}{3} \right\rceil &\leq \left\lceil \frac{4\ell+4}{12} \right\rceil \\ &\leq \left\lceil \frac{3\ell+3+\ell+1}{12} \right\rceil \\ &\leq \left\lceil \frac{3\ell+3+3i}{12} \right\rceil && (\text{since } \ell+1 \leq 3i) \\ &\leq \left\lceil \frac{\ell+1+i}{4} \right\rceil. \end{aligned}$$

In the end, we may state that

$$\text{a subpath of length } \ell \text{ hits at most } \left\lceil \frac{\ell+1+i}{4} \right\rceil \text{ vertices on a } i\text{-pattern.} \quad (2)$$

2.3 Large grids

As said before, small grids require some extra-care. In this part, we only consider grids with dimensions at least 8 in both directions. Fix n and m two even integers greater than or equal to 8. We let $k = n/2$ and $k' = m/2$. We view each side of the grid as a path from which we remove the last three vertices (see Figure 2). In these paths, we pack an adequate number of vertices using a specific i -pattern. Finally, we will estimate an upper bound on the number of such vertices in a ball of size r . This will cover most of the radii but the last few ones will be treated using some tailor-made arguments.

We use the i -patterns (where $i = 0, 1$, or 2) to select vertices on “horizontal” and “vertical” sides. The packing on the horizontal sides is as follows (with the vertical sides being similar). Our choice of i depends on the parity of k . In all cases, $2k - 4 \geq 3i$. If k is even, then we use a 1-pattern on the top $(0, m-1), (3, m-1), (7, m-1), \dots, (n-5, m-1)$ and a 1-pattern on the bottom $(n-1, 0), (n-4, 0), (n-8, 0), \dots, (4, 0)$. In this case we shall write $i_t = i_b = 1$. If k is odd, we use a 2-pattern on the top $(0, m-1), (3, m-1), (6, m-1), (10, m-1), \dots, (n-4, m-1)$ and a 0-pattern on the bottom $(n-1, 0), (n-5, 0), \dots, (5, 0)$. In this case we write $i_t = 2$ and $i_b = 0$. Using (1), we see there are exactly $\frac{k}{2}$ vertices selected on a (horizontal) side, when k is even. When k is odd $\lfloor \frac{k}{2} \rfloor$ are

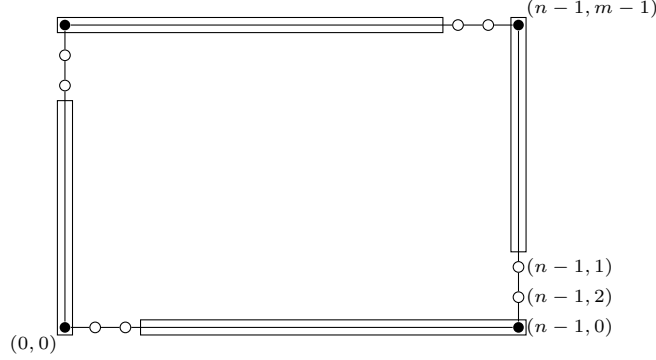


Figure 2: General sketch, packing on the perimeter.

selected on the bottom while $\lceil \frac{k}{2} \rceil$ are selected on the top. In all cases $\frac{n}{2} = k$ vertices are selected. These selections are depicted on Figure 3 for $n = 16$ and $n = 18$ (only the top and bottom sides of the grid are drawn). We call H the set of vertices selected on the horizontal paths. Similarly we select a total of k' vertices on the vertical sides and let V denote these vertices. After this process, we have a set P of $k + k'$ vertices. We shall prove that it is a valid multipacking.

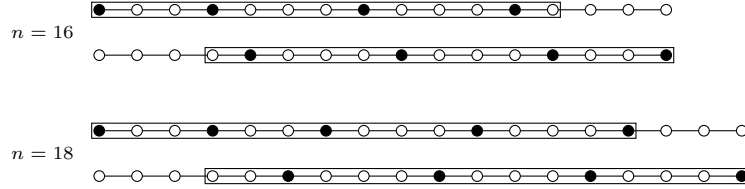


Figure 3: Selection of vertices on horizontal paths

Most balls are valid. Let r be an integer between 1 and $k + k' - 4$, and let B be a ball of radius r . If this ball does not intersect any side of the grid, then its intersection with P is empty and so $|B \cap P| \leq r$. If it intersects only one side or two consecutive sides, then its intersection lies on an isometric path of the grid where selected vertices are at distance at least 3 from each other. Thus the size of the intersection is bounded above by $\lceil \frac{2r+1}{3} \rceil$ which is at most r since r is a positive integer (see Lemma 3).

Suppose that B intersects two opposite sides of the grid (for example top and bottom). Let y denote the ordinate of its centre (bottom being ordinate 0 and top being ordinate $m - 1$). Observe that in the grids, the metric is similar to the ℓ_1 metric. Then B intersects the bottom side on a subpath of length at

most $2(r-y)$ and the top side on a subpath of length at most $2(r-2k'+1+y)$. We claim that $|B \cap H|$ is at most $r - k' + 2$. We use (2):

$$\begin{aligned} |B \cap H| &\leq \left\lceil \frac{2(r-y)+1+i_b}{4} \right\rceil + \left\lceil \frac{2(r-2k'+1+y)+1+i_t}{4} \right\rceil \\ &\leq \left\lceil \frac{2(r-y)+1+i_b}{4} \right\rceil + \left\lceil \frac{2(r+y)+3+i_t}{4} \right\rceil - k'. \end{aligned}$$

We observe that $(r-y)$ and $(r+y)$ have the same parity. Suppose $(r \pm y)$ is even and $i_b = i_t = 1$. Then $2(r-y)+1+i_b$ is 2 modulo 4 and $2(r+y)+3+i_t$ is 0 modulo 4. Thus we may remove the ceilings by adding a total of $\frac{2}{4}$. That is,

$$\begin{aligned} |B \cap H| &\leq \frac{2(r-y)+1+i_b}{4} + \frac{2}{4} + \frac{2(r+y)+3+i_t}{4} - k' \\ &= r - k' + 2 \end{aligned}$$

Using a similar analysis, we may calculate the fraction added to each term when the ceiling is removed. This is summarized below.

$r \pm y$	$i_t = i_b = 1$	$i_b = 0, i_t = 2$
even	$2/4 + 0 = 1/2$	$3/4 + 3/4 = 3/2$
odd	$0/4 + 2/4 = 1/2$	$1/4 + 1/4 = 1/2$

In three of the cases at most $1/2$ is added when the ceilings are removed. The final case $i_b = 0, i_t = 2$ and $r \pm y$ is even requires some extra care. Thus assume $r+y$ is even. We may assume the entire path of length $2(r+y)$ intersects the top row (otherwise shift the path to the right and the bound obtained will suffice for the original path). If the left most vertex is not $(0, m-1)$, then the subpath of length $2(r+y)$ intersects a 0 or 1-pattern. We can replace $i_t = 1$ and get the total added when the ceiling removed is $3/4 + 0 = 3/4$. In the case the path starts at $(0, m-1)$, we observe the right hand end of the path is at $(0, 2(r+y))$. However, the 2-pattern selects vertices in columns 3, 6, 10, 14, $\dots$. Hence, the right hand end of the path is not a selected vertex, i.e. the path contains selected vertices from a path of order at most $2(r+y)$. This gives

$$\begin{aligned} |B \cap H| &\leq \left\lceil \frac{2(r-y)+1+0}{4} \right\rceil + \left\lceil \frac{2(r-2k'+1+y)+2}{4} \right\rceil \\ &\leq \left\lceil \frac{2(r-y)+1}{4} \right\rceil + \left\lceil \frac{2(r+y)+4}{4} \right\rceil - k'. \end{aligned}$$

Again a total of $3/4 + 0 = 3/4$ is added when the ceilings are removed.

Hence, in all cases at most $3/4$ in total is added when the ceilings are removed. Using the fact that in all cases $i_b + i_t = 2$, we obtain:

$$\begin{aligned} |B \cap H| &\leq \frac{2(r-y)+1}{4} + \frac{2(r+y)+3}{4} + \frac{2}{4} + \frac{3}{4} - k' \\ &\leq r - k' + \frac{9}{4}. \end{aligned}$$

Finally, since $|B \cap H|$ is an integer, we get

$$|B \cap H| \leq r - k' + 2. \quad (3)$$

Now if B intersects at most one vertical side, this last one contributes at most $\lceil \frac{k'}{2} \rceil$ (by our choice of P). Thus

$$\begin{aligned} |B \cap P| &\leq r - k' + 2 + \left\lceil \frac{k'}{2} \right\rceil \\ &\leq r - k' + 2 + \frac{k'}{2} + \frac{1}{2} \\ &\leq r - \frac{1}{2}(k' - 5) \end{aligned}$$

which is at most r since k' is not less than 4. (When $k' = 4$ we observe $|B \cap P| \leq r + \frac{1}{2}$ implies $|B \cap P| \leq r$ since $|B \cap P|$ is an integer.)

Finally, if B intersects all four sides, we may use (3) to conclude that

$$\begin{aligned} |B \cap P| &= |B \cap H| + |B \cap V| \\ &\leq 2r - (k + k') + 4. \end{aligned}$$

This quantity is less than r whenever r is $k + k' - 4$ or less.

Balls with a big radius. To finish our proof, we only need to verify that balls with a radius r between $k + k' - 3$ and $k + k' - 1$ verify our constraint.

Let us treat the maximum radius $k + k' - 1$. Note that since n and m are both even, this grid, if seen as a chequerboard, has two diagonally opposite white corners and two diagonally opposite black corners. Suppose a ball of radius $k + k' - 1$ contains all the vertices of P . Then it must contain the four corners of the grid. Since opposite corners are at distance $2k + 2k' - 2$ it means that the centre of the ball is the middle vertex of a shortest path between opposite corners. But this middle vertex must be white for one pair of corners and must be black for the other pair, which is impossible. Thus every ball of radius $k + k' - 1$ misses at least one corner.

Now consider a ball of radius $k + k' - 2$. Since both pairs of opposite corners are at distance $2k + 2k' - 2$, at most one corner of each pair can be in a ball of such radius. Thus, such a ball misses at least two corners.

This argument can be generalized: if we find a matching of pairs in P such that each pair of vertices are at distance at least $2r + 1$, then any ball of radius r misses at least one vertex for each pair. If we match a pair of vertices at distance $2r$, then they have the same colour on the chequerboard and the centre of a ball of radius r containing both must have the same colour as the vertices when r is even and the opposite colour when r is odd.

Now we match each corner vertex with the second selected vertex on the opposite side. In the event the opposite sides are 1-patterns, the matched vertices are at distance $2k + 2k' - 5$. If one side is a 0-pattern and the other is a 2-pattern, then one pair has distance $2k + 2k' - 5$ and the other has $2k + 2k' - 6$. If three of

these corner to first selected vertex pairs are at distance $2k + 2k' - 5$, then the ball of radius $k + k' - 3$ must miss at least 3 vertices as required. Finally if we have two pairs at distance $2k + 2k' - 6$, then we observe one pair is white vertices and the other pair is black vertices. (Recall colours of the corners alternate and pairs at even distance have the same colour.) A ball of radius $k + k' - 3$ cannot contain all four vertices from these two pairs. Thus the ball misses 1 of these vertices and one each from the pairs at distance $2k + 2k' - 5$ as required.

This concludes the proof for grids with sizes at least 8 in both directions.

2.4 Long grids

The previous discussion leaves out all grids with one of their dimensions either 4 or 6. In this section, we provide a way of tackling long grids (for which $k \geq 3k' - \ell$ where ℓ depends on the parity of $k + k'$). In the end, there will only remain four cases to study.

We shall pack vertices only on the top and bottom sides of the grid. We consider the whole sides (not the $2k - 3$ first vertices as in Subsection 2.3). Recall to pack an i -pattern on a horizontal side requires $3i \leq n - 1$. If k and k' have same parity, we use a $(2k' - 3)$ -pattern on both top and bottom sides. This requires $3(2k' - 3) \leq n - 1$ or $3k' - 4 \leq k$. If k and k' have different parities, we use a $(2k' - 5)$ -pattern on one side (say bottom) and a $(2k' - 1)$ -pattern on the other. This requires $3k' - 1 \leq k$. By (1), this process selects

$$\left\lceil \frac{2k + 2k' - 3}{4} \right\rceil + \left\lceil \frac{2k + 2k' - 3}{4} \right\rceil \text{ or } \left\lceil \frac{2k + 2k' - 1}{4} \right\rceil + \left\lceil \frac{2k + 2k' - 5}{4} \right\rceil$$

vertices. In both cases, this can be simplified as $k + k'$ (in the first case, $k + k'$ is even, while it is odd in the latter).

Now, if a ball B of radius r intersects only one horizontal side of the grid, this intersection lies on an isometric path from which we selected at most every third vertex. Then by Lemma 3, it cannot contain strictly more than r vertices. Suppose that the ball B intersects both paths. Like in the previous subsection, if this ball has its centre on a vertex with ordinate y (0 being the bottom and $m - 1$ being the top), then it intersects the bottom on a path of length at most $2(r - y)$ and the top on a path of length at most $2(r - 2k' + 1 + y)$. Then we use (2). If both sides are packed with $(2k' - 3)$ -patterns,

$$\begin{aligned} |B \cap H| &\leq \left\lceil \frac{2(r - y) + 2k' - 2}{4} \right\rceil + \left\lceil \frac{2(r - 2k' + 1 + y) + 2k' - 2}{4} \right\rceil \\ &\leq \left\lceil \frac{2(r - y + k') - 2}{4} \right\rceil + \left\lceil \frac{2(r - k' + y)}{4} \right\rceil. \end{aligned}$$

And since $r - y + k'$ and $r + y - k'$ have same parity, one of the ceilings adds $\frac{1}{2}$, and

$$\begin{aligned} |B \cap H| &\leq \frac{2(r - y + k') - 2}{4} + \frac{2(r - k' + y)}{4} + \frac{1}{2} \\ &\leq r. \end{aligned}$$

Similarly, if we use the $(2k' - 5)$ -pattern on bottom and the $(2k' - 1)$ -pattern on the top, we have

$$\begin{aligned} |B \cap H| &\leq \left\lceil \frac{2(r - y) + 2k' - 4}{4} \right\rceil + \left\lceil \frac{2(r - 2k' + 1 + y) + 2k'}{4} \right\rceil \\ &\leq \left\lceil \frac{2(r - y + k') - 4}{4} \right\rceil + \left\lceil \frac{2(r - k' + y) + 2}{4} \right\rceil. \end{aligned}$$

Once again, the rounding adds at most $\frac{1}{2}$ and

$$|B \cap H| \leq r.$$

When $k' = 2$ and k is even, we use a $2k' - 3 = 1$ pattern. Thus the previous paragraph is valid for all even $k \geq 2$. When k is odd we use a 0-pattern and a 3-pattern. This requires $k \geq 5$. In particular, we have valid multipackings for $4 \times n$ for any even $n \geq 4$ and $n \neq 6$. In the same manner the previous paragraph gives a valid multipacking of order $k + k'$ when $k' = 3$ provided $k \geq 8$ for even k and $k \geq 5$ for odd k . Consequently we have packings of grids with dimensions $6 \times n$ for even $n \neq 6, 8, 12$. This concludes the proof for long grids. (We remark the above arguments show for a fixed k' and sufficiently large k , there is an optimal multipacking selecting vertices only on the horizontal sides.)

2.5 Remaining cases

Subsection 2.3 covers large grids ($4 \leq k \leq k'$), and Subsection 2.4 covers long grids ($2 \leq k' \leq (k + \ell)/3$). There are four remaining cases that can be checked by hand, and have been verified using SageMath. Three are depicted on Figure 4.

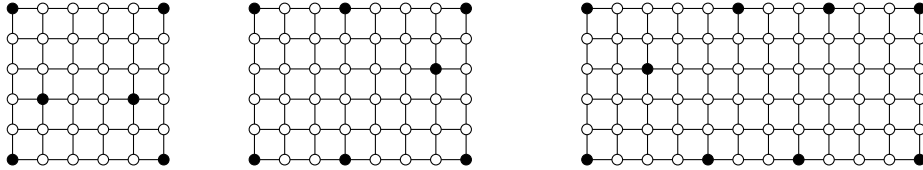


Figure 4: Multipacking for 6×6 , 8×6 , and 12×6 grids.

Finally, the 6×4 grid is the only grid with dimensions at least 4 and multipacking number strictly smaller than expected. It is 4 while its broadcast domination number is 5. This completes the proof of Theorem 1.

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