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Evaluation of failure probability under parameter epistemic uncertainty: Application to aerospace system reliability assessment

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ABSTRACT

This paper aims at comparing two different approaches to perform a reliability analysis in a context of uncertainties affecting probability distribution parameters. The first approach called "nested reliability approach" (NRA) is a classical double-loop-approach involving a sampling phase of the parameters and then a reliability analysis for each sampled parameter value. A second approach, called "augmented reliability approach" (ARA), requires to sample both distribution parameters and basic random variables conditional to them at the same phase and then integrate simultaneously over both domains. In this article, a numerical comparison is led. Possibilities offered by both approaches are investigated and the advantages of the ARA are illustrated through the application on two academic test-cases illustrating several numerical difficulties (low failure probability, nonlinearity of the limit-state function, correlation between input basic variables) and two real space system characterization (a launch vehicle stage fallback zone estimation and a collision probability between a space debris and a satellite estimation) for which only the ARA is tractable.

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1. Introduction

Reliability analysis appears to be one of the dedicated tools to quantify the risk of failure for complex aerospace systems regarding the uncertainties affecting their behavior and to help engineers to make more informed decisions in the design phase. Under safety requirements, one needs to quantify a probability of failure p_f. However, failure scenarios possibly impacting the behavior of a system often lead to rare events, i.e. events associated to a very low failure probability [1,2]. Estimating such a probability is often burdensome since classical methods such as crude Monte Carlo (CMC) involve a large number of model evaluations which make the calculations untractable [3]. Thus, the particular case of coupling between reliability analysis (underlying on multiple probabilistic analyses) and expensive aerospace computer codes is definitely a well-known issue [4].

Simulation models used in aerospace engineering became more and more complex over the last decades and reached high fidelity representation. However, they suffer from various key computer illnesses: high-dimensionality, high-nonlinearities and multi-physics combination. All in all, the common similarity between all these codes is their expensive-to-evaluate aspect. On top of that, different sources of uncertainty affect the way we evaluate the system performances (for instance, a number of input variables are uncertain). These uncertainties can be considered as inherent from nature, due to various assumptions or numerical approximations, or finally steming from measurement errors. From a pragmatic engineering point of view, uncertainties can be separated into two categories: *aleatory* and *epistemic* [5]. Aleatory uncertainty represents natural variability which is supposed to be irreducible in a specific context. Epistemic uncertainty ensues from the lack of knowledge or mathematical simplifications and can be reduced by adding more information or increasing the model fidelity. Both types of uncertainty require proper mathematical formalisms, numerical modeling and analyses. If aleatory uncertainty is often modeled using a probabilistic framework, several competing (and complementary) formalisms are available to model epistemic uncertainty: probabilistic Bayesian analysis [6], interval analysis [7], Dempster–Shafer's evidence theory [8,9], possibility theory [10], probability-boxes [11,12]. This non-exhaustive list mentions multiple frameworks which are commonly gathered under the global name of *imprecise probabilities* [13]. Mixing aleatory and epistemic uncer-

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tainties can lead to combine some of these frameworks. Choosing one of them depends on the available type of data (bounds, distribution, etc.) and has an impact on the propagation of uncertainties through the model. In this paper, we assume that probability distributions of input data are available but their parameters are not known precisely due to a lack of information or very limited data.

A classical approach, known as the "*nested reliability approach*" (NRA), is to consider sampling of uncertain distribution parameters and to perform a nested reliability analysis for each realization of these parameters (see, for example [14] for reliability assessment under epistemic uncertainty on distribution parameters given by intervals, [15] in the context of probability-based tolerance analysis of products or [16] for a coupling with Sparse Polynomial Chaos Expansions). Nonetheless, it implies to repeat several times a costly reliability analysis, which can be unaffordable for most cases in a complex industrial environment. Based on the context of complex aerospace systems design and taking into account all the constraints mentioned earlier, assessing reliability coupled with consideration on the uncertainty affecting the distribution parameters seems to be quite challenging. However, some researchers proposed methods to incorporate this kind of uncertainty affecting distribution parameters into a more general Bayesian framework [6,17–19].

An alternative approach, known as the "*augmented reliability approach*" (ARA), aims at computing a different failure probability (called "*predictive failure probability*") which takes into account the uncertainty in the distribution parameters by considering an "*augmented input space*" of the basic variables with their uncertain distribution parameters. Based on these considerations, the aim of the paper is to describe the two approaches (nested vs. augmented) within the same framework and investigate different advantages offered by ARA. Moreover, explanations are given about the key point of the transformation between the physical space of the basic input variables and the standard normal space, which is commonly used in reliability analysis, under the consideration of this new "augmented space". This article aims at giving a comparison between NRA and ARA through numerical application to challenging test-cases representing the main difficulties that aerospace engineering has to face to (nonlinear codes, correlated inputs and a low failure probability to estimate). Another goal is to highlight, through numerical results, advantages and drawbacks of both approaches coupled with advanced reliability methods tested on a benchmark representative of real world aerospace problems for which only the ARA is a tractable approach.

This paper is organized as follows. Section 2 expounds a bibliography review of reliability assessment under distribution parameter uncertainty and aims at introducing the formal concepts and notations. Section 3 defines the two approaches into a common framework and provides generic algorithms for both methods. Section 4 will illustrate the benefits of such an augmented approach through a numerical comparison between NRA and ARA on different test-cases of increasing complexity (from academic toy-cases to real black-box computer codes issued from aerospace research). Section 5 discusses limitations of those approaches and evokes possible enhancements. A conclusion gathering the most important results of this paper is finally given in Section 6.

2. Formulation of failure probability estimation under distribution parameter uncertainty

2.1. Generic time-invariant reliability problem statement

A model $\mathcal{M}(\cdot)$ is considered such that it represents a static (i.e. time is not an explicit variable here) input–output system given by: $y = \mathcal{M}(\mathbf{x})$, where $\mathbf{x} \in \mathcal{D}_{\mathbf{X}} \subseteq \mathbb{R}^d$ is a d-dimensional vector of input variables and $\mathcal{M} : \mathcal{D}_{\mathbf{X}} \to \mathcal{D}_Y \subseteq \mathbb{R}$ a given scalar mapping. In general, this mapping can be either defined using an analytical expression or a numerical model. In our case, this model is a computationally expensive simulation code which can lead to consider it as a *black-box* function only known pointwise. In the rest of the paper, $\mathcal{M}(\cdot)$ is supposed to be a deterministic model, i.e. the underlying behavior of the model is not stochastic.

The scalar random variables $X_1, X_2, \ldots, X_d$ represent the *uncertain* input variables (denoted as the *basic variables* in the following, see [20] and [21]) of the system. These basic variables are gathered in a d-dimensional random vector $\mathbf{X}$ of known continuous joint *probability density function* (pdf) $f_{\mathbf{X}} : \mathbb{R}^d \to \mathbb{R}_+$. In the space of the realizations $\mathbf{x} = (x_1, x_2, \ldots, x_d)^\top$ of the random vector $\mathbf{X}$, failure is characterized by the use of a function $g : \mathbb{R}^d \to \mathbb{R}$ called the *limit-state function* (lsf). A classical formulation for the lsf in the context of static input–output model can be:

$$g(\mathbf{X}) = y_{\text{th}} - \mathcal{M}(\mathbf{X}) \tag{1}$$

where $y_{\text{th}} \in \mathbb{R}$ is a characteristic threshold output value beyond the one the system falls into a failure state. Thus, one can distinguish two domains associated to the behavior of $g(\cdot)$: the *failure domain* given by $\mathcal{F}_{\mathbf{x}} = \{\mathbf{x} \in \mathcal{D}_{\mathbf{X}} : g(\mathbf{x}) \leq 0\}$, which in fact does include the *limit-state (hyper-)surface* (LSS) $\mathcal{F}^0_{\mathbf{x}} = \{\mathbf{x} \in \mathcal{D}_{\mathbf{X}} : g(\mathbf{x}) = 0\}$ splitting the space into two, and the *safe domain* $\mathcal{S}_{\mathbf{x}} = \{\mathbf{x} \in \mathcal{D}_{\mathbf{X}} : g(\mathbf{x}) > 0\}$. With no consideration of any distribution parameter uncertainty, the *failure probability* p_{f} therefore reads:

$$\begin{aligned} p_{\text{f}} &= \mathbb{P}[g(\mathbf{X}) \leq 0] = \int_{\mathcal{F}_{\mathbf{x}}} f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_{\mathcal{D}_{\mathbf{X}}} \mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x} \\ &= \mathbb{E}_{f_{\mathbf{X}}}\left[\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X})\right] \end{aligned} \tag{2}$$

where $\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\cdot)$ is the indicator function of the failure domain defined such that $\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) = 1$ if $\mathbf{x} \in \mathcal{F}_{\mathbf{x}}$ and $\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) = 0$ otherwise.

Finally, estimating such a failure probability can be achieved using one of the classical methods available in the structural reliability literature [22]. To do so, two distinct classes of methods have been developed: *approximation methods* such as the *First-Order Reliability Method* (FORM) and the *Second-Order Reliability Method* (SORM), which both rely on the concept of *Most-Probable Point* (MPP); and *simulation methods* based on Monte Carlo simulations [21]. Among this second class of methods, one can find more advanced sampling-based methods such as *Importance Sampling* (IS) [3], *Directional Sampling* (DS) [20], *Line Sampling* (LS) or *Subset Simulations* (SS) [4].

Approximation methods (such as FORM/SORM) have been developed, following well-argued mathematical and historical reasons [23,21], in the so-called *standard normal space* (denoted as **U**-*space*) in which all random components of $\mathbf{X}$ become independent standard Gaussian variates gathered in the vector $\mathbf{U}$. Among the simulation methods, the use of such a standard normal space is not always required (e.g., CMC method is performed in the original *physical space*, denoted as **X**-*space*). However, most advanced sampling-based methods such as those cited above are dedicated, or have some adapted versions of their initial algorithms, to the standard normal space. The general idea is to construct a regular transformation $T : \mathcal{D}_{\mathbf{X}} \to \mathbb{R}^d$ allowing (in terms of probability distributions) to get:

$$\mathbf{U} = T(\mathbf{X}) \Leftrightarrow \mathbf{X} = T^{-1}(\mathbf{U}) \tag{3}$$

where $\mathbf{U} = (U_1, U_2, \ldots, U_d)^\top$ is a d-dimensional standard Gaussian vector of independent normal variates U_i with zero means and unit standard deviations. Then, one can define a new mapping for the lsf in the standard space considering $G : \mathbb{R}^d \to \mathbb{R}$ defined such that:

$$\mathbf{U} \mapsto G(\mathbf{U}) = \left(g \circ T^{-1}\right)(\mathbf{U}) \tag{4}$$

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which allows to rewrite the failure probability:

$$p_f = \mathbb{P}[G(\mathbf{U}) \leq 0] = \int_{\mathcal{F}_\mathbf{u}} \varphi_d(\mathbf{u})\mathrm{d}\mathbf{u} = \int_{\mathbb{R}^d} \mathbb{1}_{\mathcal{F}_\mathbf{u}}(\mathbf{u})\varphi_d(\mathbf{u})\mathrm{d}\mathbf{u}$$
$$= \mathbb{E}_{\varphi_d}\left[\mathbb{1}_{\mathcal{F}_\mathbf{u}}(\mathbf{U})\right] \tag{5}$$

where $\mathcal{F}_\mathbf{u} = \{\mathbf{u} \in \mathbb{R}^d : G(\mathbf{U}) \leq 0\}$ stands for the failure domain in the standard space, $\mathrm{d}\mathbf{u} = \mathrm{d}u_1\mathrm{d}u_2\ldots\mathrm{d}u_d$ and $\varphi_d : \mathbb{R}^d \to \mathbb{R}_+$ is the d-dimensional standard Gaussian pdf of $\mathbf{U}$.

The choice of the transformation $T(\cdot)$ generally depends on the available information. When only the marginal distributions $f_{X_i}(\cdot)$ and the linear correlations are known, following the recommendations in [24,25], one should use the so-called *Nataf transformation* [26]. When the full knowledge of the joint pdf $f_\mathbf{X}(\cdot)$ is available, it is advised [27] to better use the so-called *Rosenblatt transformation* [28]. Thus, under the assumption of normal copula [29], without any consideration of parameter uncertainty, both transformations can be used since they are identical in this specific case [30].

2.2. Reliability analysis under distribution parameter uncertainty

For complex systems such as aerospace ones, the joint pdf $f_\mathbf{X}(\cdot)$ is not accurately known [17,23]. For example, the choice of a parametric model for the density $f_\mathbf{X}(\cdot)$ can be based on estimation of some distribution parameters (i.e. some moments of the pdf) which can introduce an important bias if the initial samples only provide some limited information. Moreover, some expert-judgment-based assumptions can lead to an a priori choice of some values for the parameters instead of others. The perfect knowledge of the joint pdf $f_\mathbf{X}(\cdot)$ would require, from a general point of view, the full knowledge of the marginal pdfs and the copula. However, the probabilistic information available about the input random vector $\mathbf{X}$ often reduces to the marginal distributions and, in the case of dependent inputs, to the imperfect knowledge of the linear correlation matrix $\mathbf{R} = [\rho_{ij}]_{i,j\in\{1,\ldots,d\}}$ [29]. Thus, in addition to the first uncertainty level characterizing the basic input variables, uncertainty may also affect both distribution parameters and the dependence structure. Consequently, engineers have to face what we call a *bi-level uncertainty*. In this context, giving back an hypothetic single measure of reliability taking only one level into account seems to be inappropriate.

Such a topic has been early discussed among the structural reliability community, mainly in the first investigations led by Der Kiureghian [6,17,31,32] and Ditlevsen [18–20,33]. In their common paper [5], these authors stress the need of a measure of reliability that takes into account parameter uncertainty (the first author proposed in [6] to call it "*predictive reliability measure*", following Bayesian analysis vocabulary, and provided a formal definition that will be recalled later).

Assuming now that $\mathbf{X}$ is distributed according to the parametric joint pdf $f_{\mathbf{X}|\boldsymbol{\Theta}}(\cdot|\cdot)$, each random variable X_i is distributed according to the marginal pdf $f_{X_i|\boldsymbol{\Theta}_i}(\cdot|\cdot)$. In the case of dependent inputs, in the normal copula case, uncertainty affecting the correlation matrix could easily be considered in this framework. However, from a more general point of view, uncertainty affecting the dependence structure (i.e. the copula) is not a widely studied topic in literature. Moreover, from an engineering perspective, this problem is really difficult to assess due to the crucial lack of information. In this paper, we will only consider distribution parameter uncertainty and let copula structure uncertainty to future work. Indeed the vector $\boldsymbol{\Theta}$ gathers all distribution parameters of the corresponding marginals such that $\boldsymbol{\Theta} = (\boldsymbol{\Theta}_1, \boldsymbol{\Theta}_2, \ldots, \boldsymbol{\Theta}_d)^\top$, where each $\boldsymbol{\Theta}_i, i \in [\![1, d]\!]$ is a set of distribution parameters for the i-th marginal (for instance, if $X_i \sim \mathcal{N}(\mu_{X_i}, \sigma_{X_i})$, then $\boldsymbol{\Theta}_i = (\mu_{X_i}, \sigma_{X_i})^\top$). One can imagine that depending on the distribution type, all the marginal pdfs will not be defined with the same number of parameters. In this paper, we assume that only a set of uncorrelated distribution parameters are uncertain which leads to consider a general collection of univariate random parameters given by $\boldsymbol{\Theta} = (\Theta_1, \Theta_2, \ldots, \Theta_k)^\top \in \mathcal{D}_\boldsymbol{\Theta} \subseteq \mathbb{R}^k$ (which can be either moments or bounds). Consequently, without any loss of generality, one can assume the existence of a joint pdf $f_\boldsymbol{\Theta} = \prod_{j=1}^k f_{\Theta_j}$ as a product of the marginal pdfs of the Θ_j [34]. Note here that one could also consider a dependence structure between the distribution parameters. However, the problem would be far more difficult and would imply to have, at minimum, a prior information about such a dependence structure. This topic is beyond the scope of this paper. To get a deeper insight about the practical characterization of $f_\boldsymbol{\Theta}(\boldsymbol{\theta})$ based on available data (which is not the scope of this paper), the reader may refer to [35]. To sum up, in this paper, only a *prior probability distribution* (for instance, following an expert-based judgment) will be assumed for $\boldsymbol{\Theta}$ without any purpose of Bayesian reliability updating [36].

Thus, a new formulation for the failure probability can be proposed, following [17]. Indeed, due to this bi-level uncertainty (on the vector of basic variables $\mathbf{X}$ and on the vector of distribution parameters $\boldsymbol{\Theta}$), the failure probability p_f is no more a deterministic value. It becomes a random variable, denoted as P_f, which depends on the realization $\boldsymbol{\theta}$ of the random vector of uncertain parameters such that:

$$P_f(\boldsymbol{\theta}) = \mathbb{P}[g(\mathbf{X}) \leq 0 \mid \boldsymbol{\Theta} = \boldsymbol{\theta}] \tag{6a}$$
$$= \int_{\mathcal{D}_\mathbf{X}} \mathbb{1}_{\mathcal{F}_\mathbf{x}}(\mathbf{x}) f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x}|\boldsymbol{\theta}\mathrm{d}\mathbf{x} \tag{6b}$$
$$= \mathbb{E}_{f_{\mathbf{X}|\boldsymbol{\Theta}}}\left[\mathbb{1}_{\mathcal{F}_\mathbf{x}}(\mathbf{X}) \mid \boldsymbol{\Theta} = \boldsymbol{\theta}\right]. \tag{6c}$$

Hence, by integrating over $\boldsymbol{\theta}$, we get the so-called "*predictive failure probability*" $\widetilde{P}_f$ which is a measure of reliability taking into account the effect of the uncertain characterization of distribution parameters:

$$\widetilde{P}_f = \mathbb{E}_{f_\boldsymbol{\Theta}}[P_f(\boldsymbol{\Theta})] \tag{7a}$$
$$= \int_{\mathcal{D}_\boldsymbol{\Theta}} P_f(\boldsymbol{\theta}) f_\boldsymbol{\Theta}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta} \tag{7b}$$
$$= \int_{\mathcal{D}_\boldsymbol{\Theta}} \left(\int_{\mathcal{D}_\mathbf{X}} \mathbb{1}_{\mathcal{F}_\mathbf{x}}(\mathbf{x}) f_{\mathbf{X}|\boldsymbol{\Theta}}(\mathbf{x}|\boldsymbol{\theta})\mathrm{d}\mathbf{x} \right) f_\boldsymbol{\Theta}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}. \tag{7c}$$

Eq. (7c) is the key equation whose solving is under consideration in this paper. The idea is that it can be numerically solved by two different approaches.

From a numerical point of view, a first way of computing this integral relies on evaluating pointwise the inner integral for each realization $\boldsymbol{\theta}$ of $\boldsymbol{\Theta}$ [14–16]: this leads to the nested reliability approach (presented in subsection 3.1). The second way consists in evaluating it by treating both basic variables and uncertain distribution parameters together and by integrating simultaneously on both domains (but still respecting the conditioning) as suggested in [17]: this is the augmented reliability approach (presented in subsection 3.2). The next section describes these approaches in details. As a remark, one can notice that this Bayesian framework provides here a single reliability measure (the predictive failure probability). Nevertheless, this quantity can help engineers to make more informed decisions during the design process and can be coupled with the classical reliability measure so as to analyze properly the risk undertaken with a design choice. Decision can be then enlightened by such additional information [37–39].

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Up to now, to our knowledge, several researchers deployed efforts to carry on the way of other approaches to compute this predictive failure probability (see another approach by [40], used in [6] and in [41] only with FORM calculations). Nevertheless, the track of exploring the *augmented space* has not been over-exploited yet. In [42], the author recommended and implemented this strategy on a fracture mechanics test-case but limited his study to FORM algorithm. All these works mainly focused on providing a global reliability index, robust to parameter uncertainty, in the specific context of FORM. The use of an augmented space has also been exploited by [43] for design sensitivity purpose while considering uncertain design parameters. More recently, in [44], the authors proposed in a broader view on the comprehension and interpretation of the different levels of uncertainty involved in these calculations and encouraged to use an augmented approach to solve a similar integral problem given in Eq. (7c). However, their study did not aim at performing reliability assessment for rare event failure probabilities of some complex simulation codes which is the scope of the present paper.

3. Description of the two approaches

3.1. The nested reliability approach (NRA)

This approach is a nested-loop-based approach since it involves the numerical estimation of two different quantities. The first (nested or inside) loop aims at computing a "conditional" failure probability whose numerical estimator is denoted as $\widehat{P}_{\mathrm{f}}(\boldsymbol{\theta})$. This estimator is a measure of reliability under the realization $\boldsymbol{\theta}$ of the random vector $\boldsymbol{\Theta}$. The second (outside) loop aims at computing an estimator of the predictive failure probability, denoted as $\widehat{\widehat{P}}_{\mathrm{f}}$, by integrating over the support of the random vector $\boldsymbol{\Theta}$. In practice, it consists in computing several $\widehat{P}_{\mathrm{f}}(\boldsymbol{\theta})$ for a range of realizations $\boldsymbol{\theta}$ of the vector of uncertain parameters $\boldsymbol{\Theta}$. It has been widely used in literature, in various contexts, such as rare event probability estimation with Kriging-based approach in [14], probability-based tolerance analysis of products in [15] or uncertainty propagation using probability-boxes and polynomial chaos expansions in [16].

Algorithm 1 Nested reliability approach (NRA) with CMC for probability estimation.

1: **Start**
2: **Define**: $\mathbf{X}$, $\boldsymbol{\Theta}$, $N_{\mathbf{x}}$ and $N_{\boldsymbol{\theta}}$ (resp. the sampling budgets over the domain $\mathcal{D}_{\mathbf{X}}$ and the domain $\mathcal{D}_{\boldsymbol{\Theta}}$)
3: **For** $k = 1 : N_{\boldsymbol{\theta}}$
4: Sample $\boldsymbol{\Theta}^{(k)}$;
5: **For** $i = 1 : N_{\mathbf{x}}$
6: Sample $\mathbf{X}^{(i)}$ given $\boldsymbol{\Theta}^{(k)} = \boldsymbol{\theta}^{(k)}$;
7: Limit-state function evaluation: $g(\mathbf{x}^{(i)})$;
8: **Conditional failure probability estimator**:
9: $\widehat{P}_{\mathrm{f}}^{(k)}(\boldsymbol{\theta}^{(k)}) = \frac{1}{N_{\mathbf{x}}} \sum_{i=1}^{N_{\mathbf{x}}} \mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X}^{(i)})$;
10: **Predictive failure probability estimator**:
11: $\widehat{\widehat{P}}_{\mathrm{f}} = \frac{1}{N_{\boldsymbol{\theta}}} \sum_{k=1}^{N_{\boldsymbol{\theta}}} \widehat{P}_{\mathrm{f}}^{(k)}(\boldsymbol{\theta}^{(k)})$;
12: **End**

A generic implementation of NRA framework coupled with a nested CMC method is given in the Algorithm 1. In the rectangular box at lines 5–9, one can choose any available reliability method to estimate the conditional failure probability $\widehat{P}_{\mathrm{f}}(\boldsymbol{\theta})$, from approximation methods (FORM, SORM) to most advanced simulation methods (IS, SS). Nevertheless, it seems more relevant to focus on the ones that are still, up-to-now, the most widely used either in aerospace industry or in research. Indeed CMC is still considered as the reference method for validation. FORM offers wide possibilities for practitioners who want to perform reliability assessment with a low computational cost, even if this method only gives the true failure probability for linear lsfs [21]. Finally,

Algorithm 2 NRA generic box (FORM or SS).

...
Define: classical transformation T to the **U**-space
$\mathbf{u} = T(\mathbf{x})$ (Nataf or Rosenblatt) and its inverse
Start
FORM Algorithm / SS Algorithm
(see [4] for a full description of these algorithms)
End
Get: $\widehat{P}_{\mathrm{f}}^{(k)}(\boldsymbol{\theta}^{(k)})$
...

SS appeared to be a very powerful method to reach estimation of rare event failure probabilities, under the constraint of nonlinear lsfs, with a rather moderate computational effort [45]. In brief, the rectangular box can be seen as a non-intrusive plug-in uncertainty propagation code for reliability assessment. An example of a plug-in box (for FORM or SS) is given in the Algorithm 2. In numerous cases, an additional step is required: the transformation to the standard normal space (see subsection 2.1). In the nested case, the transformation is already included in the plug-in reliability rectangular box, i.e. classical transformations such as Nataf or Rosenblatt ones can be both used, and the distribution parameter uncertainty does not change anything to their implementation. Nevertheless, one should notice that for each sampled parameter, the algorithm needs to rebuild and recalculate the transformation since it depends on the parameter value. Thus, for complicated transformations, with a large number of basic variables, the simulation cost induced can be increased.

In this nested case, we can demonstrate that the estimator $\widehat{\widehat{P}}_{\mathrm{f}}$ is unbiased. Moreover, the mean and variance of $\widehat{\widehat{P}}_{\mathrm{f}}$ are estimated by replication of the algorithm, using the following classical statistics:

$$m_{\widehat{\widehat{P}}_{\mathrm{f}}} = \frac{1}{N_{rep}} \sum_{i=1}^{N_{rep}} \widehat{\widehat{P}}_{\mathrm{f}}^{(i)} \tag{8}$$

which is the *sample mean* with N_{rep} the number of replications of the predictive failure probability estimation and $S^2_{\widehat{\widehat{P}}_{\mathrm{f}}}$ the *unbiased sample variance* defined by:

$$S^2_{\widehat{\widehat{P}}_{\mathrm{f}}} = \frac{1}{N_{rep} - 1} \sum_{i=1}^{N_{rep}} \left(\widehat{\widehat{P}}_{\mathrm{f}}^{(i)} - m_{\widehat{\widehat{P}}_{\mathrm{f}}} \right)^2 . \tag{9}$$

3.2. The augmented reliability approach (ARA)

Another approach is to consider an *augmented input random vector* $\mathbf{Z} \overset{\text{def}}{=} (\boldsymbol{\Theta}, \mathbf{X})^{\top}$ composed of the basic variables and their distribution parameters as it appears in Eq. (7c) (see [16] for a similar definition). Thus, this augmented input space has a dimension of $k + d$ (k uncertain distribution parameters Θ_j and d random basic variables X_i).

A generic implementation framework is given in the Algorithm 3. Again, in this algorithm, the rectangular box can be replaced by any non-intrusive plug-in uncertainty propagation code for reliability assessment as the ones cited previously for the NRA (see Algorithm 4 as an example). This shows that the ARA does not suffer from any major difference with the classical nested approach in terms of the variety of methods that it can handle. Again, as for the NRA, one can demonstrate that the estimator $\widehat{\widehat{P}}_{\mathrm{f}}$ is unbiased. One major difference concerns the transformation to standard normal space: since there exists a conditioning between the

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Algorithm 3 Augmented reliability approach (ARA) bluewith CMC for probability estimation.

Start
Define: $\mathbf{X}$, $\boldsymbol{\Theta}$, $N_{\mathbf{x},\boldsymbol{\theta}}$ (sampling budget)
For $i = 1 : N_{\mathbf{x},\boldsymbol{\theta}}$
  Sample $\boldsymbol{\Theta}^{(i)}$;
  Sample $\mathbf{X}^{(i)}$ given $\boldsymbol{\Theta}^{(i)} = \boldsymbol{\theta}^{(i)}$;
  Limit-state function evaluation: $g(\mathbf{x}^{(i)})$;
Predictive failure probability estimator:
$\widehat{\widetilde{P}}_{\mathrm{f}} = \frac{1}{N_{\mathbf{x},\boldsymbol{\theta}}} \sum_{i=1}^{N_{\mathbf{x},\boldsymbol{\theta}}} \mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X}^{(i)})$;
End

Algorithm 4 ARA generic box (FORM or SS).

...
Define: new transformation T for the augmented space (Rosenblatt on the augmented input vector)
$\mathbf{u} = T(\mathbf{z})$
Start
  FORM Algorithm / SS Algorithm
End
Get: $\widehat{\widetilde{P}}_{\mathrm{f}}$
...

distribution parameters and the basic input variables, Nataf transformation cannot be used anymore and Rosenblatt transformation is the only one that can handle this constraint.

Considering uncertainties affecting distribution parameters leads to adapt the usual Rosenblatt transformation. We assume the joint pdf $f_{\mathbf{X}|\boldsymbol{\Theta}}(\cdot|\cdot)$ is known since we know all the marginal pdfs and the correlation matrix (or the covariance matrix) giving the linear correlation structure between the basic input variables (normal or Gaussian copula case [29]). In addition, we know the joint pdf $f_{\boldsymbol{\Theta}}(\cdot)$ as explained previously in subsection 2.2. In this case, under the consideration of the *augmented space* of dimension $k+d$ (k distribution parameters and d basic variables), one can apply Rosenblatt transformation [28] first to the k components of $\boldsymbol{\Theta}$ and then to the d components of the vector $\mathbf{X}|\boldsymbol{\Theta}$ such that:

$$T_{aug}^{Ros}: \left| \begin{array}{ccc} \mathbb{R}^{k+d} & \longrightarrow & \mathbb{R}^{k+d} \\ \mathbf{z} & \longmapsto & \mathbf{u} = \begin{pmatrix} \Phi^{-1}\left(F_{\Theta_1}(\theta_1)\right) \\ \vdots \\ \Phi^{-1}\left(F_{\Theta_k}(\theta_k)\right) \\ \vdots \\ \Phi^{-1}\left(F_{X_1|\Theta_1,\ldots,\Theta_k}(x_1|\theta_1,\ldots,\theta_k)\right) \\ \vdots \\ \Phi^{-1}\left(F_{X_d|\Theta_1,\ldots,\Theta_k,X_1,\ldots,X_{d-1}}(x_d|\theta_1,\ldots,\theta_k,x_1,\ldots,x_{d-1})\right) \end{pmatrix} \end{array} \right. \tag{10}$$

where $\Phi^{-1}(\cdot)$ is the normal inverse *cumulative density function* (CDF) and $F_{\Theta_i}(\cdot)$, $F_{X_j|\Theta_i}(\cdot|\cdot)$ respectively the marginal CDFs of the parameters and the conditional marginal CDFs of the basic variables. In the case of correlated inputs, one can implement this regular transformation following and adapting general formulas given in [28]. As a remark, one should notice that from a numerical point of view, the inverse transformation $\left(T_{aug}^{Ros}\right)^{-1}(\cdot)$ (from the standard normal space to the physical space) can be the most useful (especially when FORM, SORM or SS methods are used).

3.3. Illustration

From a numerical point of view, NRA and ARA can be illustrated through a two-dimensional example with one uncertain distribution parameter. Let us call X_1 and X_2 the two basic input variables modeled as two Gaussian variates such that $X_1 \sim \mathcal{N}(\mu_{X_1} = 7, \sigma_{X_1} = 5/\sqrt{3})$ and $X_2 \sim \mathcal{N}(\Theta = \mu_{X_2}, \sigma_{X_2} = 2/\sqrt{3})$. The mean of X_2 is considered as being uncertain (for example, $\Theta \sim \mathcal{N}(2, 1.5)$). For the sake of clarity, in Fig. 1(a), only three clouds of samples are plotted for three different values of θ (200 points per cloud). Indeed such a sequential sampling is the underlying principle of NRA.

As for ARA, graphical results plotted in Fig. 1(b) bring out the underlying principle of this approach: covering in one algorithm step the augmented input space (for the sake of comparison, 600 points are used, instead of 3×200 points for NRA). One can clearly notice the same trend between NRA and ARA, the first one by a sequential sampling strategy, the second one by a simultaneous sampling over all the dimensions of the augmented input space. In brief, it appears that ARA offers better space-filling properties than NRA [46].

4. Numerical comparison between NRA and ARA

To evaluate the efficiency of the ARA, a numerical validation benchmark has been performed with a systematic comparison to the classical NRA. Several test-cases of increasing complexity have been chosen (from two analytical cases to two different "real life" aerospace industrial applications) to check the validity of the methods. The choice of these test-cases aims at covering a range of classical problems encountered in reliability assessment of aerospace systems: complex black-box models with numerous input variables, high nonlinearities, high computational cost, low probability of failure. Moreover, three reliability methods have been tested to calculate the failure probability: CMC, FORM and SS. The following numerical applications have been implemented in Matlab® and performed using the open source toolbox FERUM v4.1 [47].

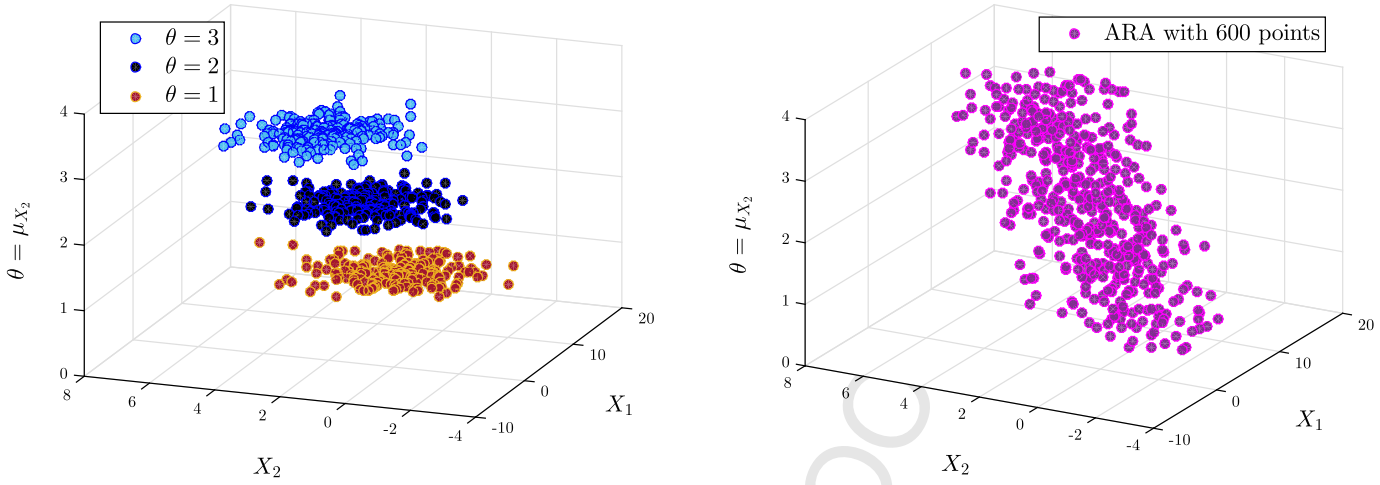
4.1. Methodology and comparison metrics

This paper aims at comparing results obtained for both NRA and ARA. For each type of approach, two reliability methods, FORM (when lsf is linear) and SS, will be used to estimate the predictive failure probability $\widetilde{P}_{\mathrm{f}}$. These combined approaches (NRA/FORM, NRA/SS, ARA/FORM, ARA/SS) will be respectively compared to a reference estimation performed using CMC (most of the time, a NRA/CMC with a large number of samples on both domains). Table 1 gives a brief overview of the methodology.

In Table 1, the black squares ■ stand for successful calculations of the test-cases and the crosses × indicate that FORM is clearly inappropriate since the lsf is known explicitly to be nonlinear. As a remark, one can notice that some specific cases are denoted as computationally "untractable". Indeed, to overcome such a difficulty and to get a reference result to make the comparison viable, specific computational strategies have been set up. For the sake of clarity and to avoid any confusion, these strategies are presented and discussed in the dedicated subsections of the test-cases.

One needs to introduce the comparison metrics used in the following numerical benchmarks. Thus, following [4], we choose in this paper to characterize the quality of our estimator $\widehat{\widetilde{P}}_{\mathrm{f}}^{M}$ of $\widetilde{P}_{\mathrm{f}}$ obtained with the method M by the use of three performance metrics computed with respect to the reference calculations, i.e. those obtained by CMC:

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(a) NRA (200 points/cloud, indeed 600 points in total). (b) ARA (600 points in the augmented space).

Fig. 1. Illustration of NRA and ARA simulation procedures on a two-dimensional problem.

Table 1
Overall methodology.

Test-case		Ref.	NRA			ARA		
		Ref. CMC	CMC	FORM	SS	CMC	FORM	SS
Correlated $R-S$[a]	(*cf.* 4.2.1)	■	■	■	■	■	■	■
Nonlinear oscillator[b]	(*cf.* 4.2.2)	■	■	×	■	■	×	■
Launch vehicle fallback zone[c]	(*cf.* 4.3)	untractable	■	■	■	■	■	■
Collision between orbiting objects[d]	(*cf.* 4.4)	untractable	untractable	■	untractable	■	■	■

[a] 2 correlated basic variables, 1 uncertain parameter, $g(\cdot)$ linear, low classical failure probability.
[b] 8 independent basic variables, 1 uncertain parameter, $g(\cdot)$ nonlinear, low classical failure probability.
[c] 6 independent basic variables, 2 uncertain parameters, $g(\cdot)$ nonlinear, low classical failure probability.
[d] 6 correlated basic variables, 3 uncertain parameters, $g(\cdot)$ nonlinear, low classical failure probability.

- the *Relative standard Error* (RE):

$$\mathrm{RE}\left[\widehat{\widehat{P}}_{\mathrm{f}}^{M}\right] = \frac{\sqrt{\mathbb{V}\left[\widehat{\widehat{P}}_{\mathrm{f}}^{M}\right]}}{\mathbb{E}\left[\widehat{\widehat{P}}_{\mathrm{f}}^{M}\right]} \tag{11}$$

where $\mathbb{E}\left[\widehat{\widehat{P}}_{\mathrm{f}}^{M}\right]$ and $\mathbb{V}\left[\widehat{\widehat{P}}_{\mathrm{f}}^{M}\right]$ are estimated on the sample obtained by replication of the analysis (see Eq. (8) and Eq. (9));

- the *Relative Bias* (RB):

$$\mathrm{RB}\left[\widehat{\widehat{P}}_{\mathrm{f}}^{M}\right] = \frac{\mathbb{E}\left[\widehat{\widehat{P}}_{\mathrm{f}}^{M}\right] - \widehat{\widehat{P}}_{\mathrm{f}}^{CMC}}{\widehat{\widehat{P}}_{\mathrm{f}}^{CMC}}. \tag{12}$$

It gives a description of how close the estimate $\widehat{\widehat{P}}_{\mathrm{f}}^{M}$ is close to the reference value $\widehat{\widehat{P}}_{\mathrm{f}}^{CMC}$. In the following (see Tables 3 to 6), RB for NRA is computed with reference to the quantity $\widehat{\widehat{P}}_{\mathrm{f,ref}}$ (estimated by a reference CMC or another method when CMC is untractable) while RB for ARA is computed with reference to $\widehat{\widehat{P}}_{\mathrm{f}}^{ARA/CMC}$ to make the comparison representative;

- the *efficiency* ν^{M} relatively to CMC estimate (respectively NRA/CMC and ARA/CMC) defined such that:

$$\nu^{M} = \frac{N_{sim}^{CMC}}{N_{sim}^{M}} \tag{13}$$

where N_{sim}^{CMC} is the required number of CMC simulations to get $\mathrm{RE}\left[\widehat{\widehat{P}}_{\mathrm{f}}^{CMC}\right] = \mathrm{RE}\left[\widehat{\widehat{P}}_{\mathrm{f}}^{M}\right]$. Thus, the ν^{M} ratio can be rewritten as:

$$\nu^{M} = \frac{\left(1-\widehat{\widehat{P}}_{\mathrm{f}}^{M}\right)}{N_{sim}^{M} \times \widehat{\widehat{P}}_{\mathrm{f}}^{M} \times \mathrm{RE}\left[\widehat{\widehat{P}}_{\mathrm{f}}^{M}\right]^{2}}. \tag{14}$$

A value of $\nu^{M} > 1$ indicates that the method M is more efficient than CMC for the given test-case. In other words, ν^{M} indicates the quantity by which we can divide the initial CMC simulation budget for a same level of accuracy.

4.2. Application on two academic test-cases

4.2.1. A first $R-S$ example with correlated basic variables and low failure probability

Description. The aim of this first academic test-case is to check the validity of the two approaches regarding two difficulties: assuming a strong correlation in the input probabilistic model and trying to estimate a low failure probability with respect to a given simulation budget. Table 2 gives the input data. The reference failure probability without parameter uncertainty is $p_{\mathrm{f,ref}} = 8.84 \times 10^{-8}$ (because of the linear lsf, the true failure probability can be obtained using FORM). The correlation coefficient $\rho = 0.9$ expresses the linear correlation between the two basic variables. The failure is considered when the sollicitation S overcomes the resource R. The lsf thus reads:

$$g(\mathbf{X}) = R - S = X_1 - X_2. \tag{15}$$

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Table 2
Second-order statistics and distributions of input random variables for the $R - S$ test-case.

Variable X_i[a]	Distribution	Mean μ_{X_i}	Std σ_{X_i}
$X_1 = R$	Normal	12	$5/\sqrt{3}$
$X_2 = S$	Normal	μ_{X_2} uncertain[b]	$2/\sqrt{3}$
$\Theta = \mu_{X_2}$	Normal	2	1.5

[a] Linear correlation between X_1 and X_2: $\rho = 0.9$.
[b] For a fixed value $\mu_{X_2} = 2$, $p_{\text{f,ref}} = 8.84 \times 10^{-8}$.

The conditional failure probability, i.e. $P_{\text{f}}(\Theta = \theta)$, can be written in its integral form since the joint conditional pdf $f_{\mathbf{X}|\Theta}(\cdot|\cdot)$ can be analytically derived. One gets:

$$P_{\text{f}}(\theta = \mu_{X_2}) = \int_{\mathcal{F}_{\mathbf{x}}} \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}} \times \exp\left[-\frac{1}{2(1-\rho^2)} \times \left(\frac{(x_1-\mu_{X_1})^2}{\sigma_{X_1}^2} - \frac{2\rho(x_1-\mu_{X_1})(x_2-\theta)}{\sigma_{X_1}\sigma_{X_2}} + \frac{(x_2-\theta)^2}{\sigma_{X_2}^2} \right) \right] \mathrm{d}\mathbf{x} . \tag{16}$$

In the specific case of two correlated normal variables and a linear lsf, the probability of failure becomes:

$$P_{\text{f}}(\theta = \mu_{X_2}) = \Phi(-\beta_C) = \Phi\left(-\frac{\mu_{X_1} - \theta}{\sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2 - 2\rho\sigma_{X_1}\sigma_{X_2}}} \right), \tag{17}$$

where β_C is the *Cornell reliability index* [48]. This simple closed-form solution can be used to check and validate numerical results obtained for this elementary test-case.

Results. Table 3 illustrates that NRA and ARA give similar results for estimating the predictive failure probability. Moreover, for almost all the methods (except NRA/CMC which suffers here from a lack of points while computing the integral over $\mathcal{D}_{\boldsymbol{\Theta}}$) it demonstrates that ARA can handle both rare event probabilities and strong correlation between basic input variables. On the one hand, ARA/FORM seriously challenges other methods since it has a very small number of simulation code evaluations compared to CMC and SS and it gives exact results since the lsf is linear. On the other hand, ARA/SS definitely gives promising results compared to ARA/CMC since the ν value ($\nu = 54.44$) is high. In a classical context of rare event (often encountered in aerospace engineering), one can see the superiority of ARA (coupled with FORM or SS) compared to other NRA-coupled methods. This test-case serves as a preliminary verification for ARA before testing it on a real aerospace simulation code such as those presented in subsections 4.3 and 4.4. It also reveals how high can be the variations between the classical failure probability estimate and the predictive one considering parameters uncertainty (here, it drops from 10^8 to 10^5).

4.2.2. A nonlinear oscillator

Description. This nonlinear oscillator is a well-known structural reliability test-case firstly proposed in [49] and then used for benchmarking purposes in [2,50,51]. The aim here, is to assess reliability of a two-degree-of-freedom primary-secondary system, as shown in Fig. 2, under a white noise base acceleration. The basic variables characterizing the physical behavior are the masses m_p and m_s, spring stiffnesses k_p and k_s, natural frequencies $\omega_p = (k_p/m_p)^{1/2}$

Table 3
Results for the $R - S$ test-case.

Approach	CMC[a]			FORM (exact)			SS[b]			
	$m_{\widehat{P}_{\text{f}}}$	$S^2_{\widehat{P}_{\text{f}}}$	RE	$m_{\widehat{P}_{\text{f}}}$	$S^2_{\widehat{P}_{\text{f}}}$	RE	RB	ν	$m_{\widehat{P}_{\text{f}}}$	$S^2_{\widehat{P}_{\text{f}}}$	RE	RB	ν
NRA[c]	2.57×10^{-5}	2.79×10^{-11}	0.21	1.99×10^{-5}	7.38×10^{-11}	0.43	-2.10×10^{-3}	–	1.99×10^{-5}	5.68×10^{-11}	0.38	-1.99×10^{-3}	48.03
ARA	1.95×10^{-5}	1.89×10^{-12}	7.03×10^{-2}	1.97×10^{-5}	–	–	7.84×10^{-3}	–	1.99×10^{-5}	7.26×10^{-11}	0.43	1.91×10^{-2}	54.44

[a] NRA: $N_{\boldsymbol{\theta}} = 10^3$ samples, $N_{\mathbf{x}} = 10^3$ samples | ARA: $N_{\mathbf{x},\boldsymbol{\theta}} = 10^6$ samples.
[b] NRA: $N_{\boldsymbol{\theta}} = 10^3$ samples, $N_{\mathbf{x}} = 10^3$ samples/step | ARA: $N_{\mathbf{x},\boldsymbol{\theta}} = 10^3$ samples/step.
[c] Ref. (NRA/CMC, $N_{\boldsymbol{\theta}} = 10^6$ samples, $N_{\mathbf{x}} = 10^6$ samples): $\widehat{P}_{\text{f,ref}} = 1.99 \times 10^{-5}$.

Table 4
Results for the nonlinear oscillator test-case.

Approach	CMC[a]			SS[b]			
	$m_{\widehat{P}_{\text{f}}}$	$S^2_{\widehat{P}_{\text{f}}}$	RE	$m_{\widehat{P}_{\text{f}}}$	$S^2_{\widehat{P}_{\text{f}}}$	RE	RB	ν
NRA[c]	1.41×10^{-4}	1.41×10^{-10}	8.39×10^{-2}	1.59×10^{-4}	2.18×10^{-10}	9.27×10^{-2}	2.77×10^{-2}	0.18
ARA	1.53×10^{-4}	1.51×10^{-10}	8.00×10^{-2}	1.52×10^{-4}	3.31×10^{-9}	0.38	-1.08×10^{-2}	11.5

[a] NRA: $N_{\boldsymbol{\theta}} = 10^3$ samples, $N_{\mathbf{x}} = 10^3$ samples | ARA: $N_{\mathbf{x},\boldsymbol{\theta}} = 10^6$ samples.
[b] NRA: $N_{\boldsymbol{\theta}} = 10^3$ samples, $N_{\mathbf{x}} = 10^3$ samples/step | ARA: $N_{\mathbf{x},\boldsymbol{\theta}} = 10^3$ samples/step.
[c] Ref. (NRA/CMC, $N_{\boldsymbol{\theta}} = 10^6$ samples, $N_{\mathbf{x}} = 10^6$ samples): $\widehat{P}_{\text{f,ref}} = 1.55 \times 10^{-4}$.

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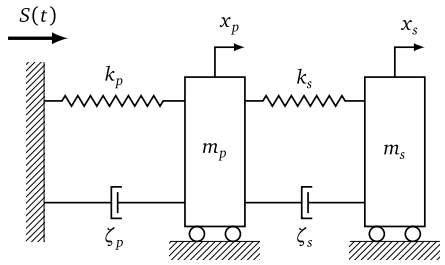


Fig. 2. Two-degree-of-freedom damped oscillator with primary and secondary systems.

and $\omega_s = (k_s/m_s)^{1/2}$ and damping ratios ζ_p and ζ_s, where the subscripts p and s respectively refer to the primary and secondary oscillators. If F_s denotes the force capacity of the secondary spring, then the reliability of the system can be evaluated using the following lsf [49,52]:

$$g(\mathbf{X}) = F_s - 3k_s \times \sqrt{\frac{\pi S_0}{4\zeta_s\omega_s^3}\left[\frac{\zeta_a\zeta_s}{\zeta_p\zeta_s\left(4\zeta_a^2+r^2\right)+\gamma\zeta_a^2}\frac{\left(\zeta_p\omega_p^3+\zeta_s\omega_s^3\right)\omega_p}{4\zeta\omega_a^4}\right]} \tag{18}$$

where S_0 is the intensity of the white noise, $\gamma = m_s/m_p$ the mass ratio, $\omega_a = (\omega_p + \omega_s)/2$ the average frequency ratio, $\zeta_a = (\zeta_p + \zeta_s)/2$ the average damping ratio and $r = (\omega_p - \omega_s)/\omega_a$ a tuning parameter. The probabilistic model for $\mathbf{X}$ is detailed in Table 7.

The two interesting characteristics of this application test-case are its set of non-normal basic random variables and the fact that it suffers from a highly nonlinear limit-state surface [50] (which prevents from using any FORM-based approach). Moreover, following [51], it seems relevant to consider the mean of the force capacity μ_{X_7} as the most influent distribution parameter on the failure probability. The nominal value for μ_{X_7} is chosen to be 21.5 N so as to reach a reference probability without parameter uncertainty $p_{\mathrm{f,ref}}$ equal to 4.75×10^{-5} [51].

Results. Numerical results summarized in Table 7 show that, for the same simulation budget, ARA/CMC is more accurate than NRA/CMC to estimate the predictive failure probability (the reference result $\widehat{\widetilde{P}}_{\mathrm{f,ref}}$ is provided below the table). As for ARA/SS, a significant gain is noticeable referring to the high ν values compared to unity ($\nu > 10$). In brief, that means the ARA/SS is very efficient to treat this problem compared to a classical Monte Carlo approach. A final remark concerns the comparison between the two reference probabilities $p_{\mathrm{f,ref}} = 4.75 \times 10^{-5}$ and $\widehat{\widetilde{P}}_{\mathrm{f,ref}} = 1.55 \times 10^{-4}$: one can see that, in this case, considering uncertainty on a distribution parameter makes the system less safe, which can be, for example, an important indicator for design or re-design purposes.

4.3. Application on a launch vehicle stage fallback zone estimation

Description. Space launcher complexity arises from the coupling between several subsystems such as stages or boosters and other embedded systems. Optimal trajectory assessment is a key discipline since it is one of the cornerstones of the mission success (for ascent as well as for re-entry trajectories). However, during the real flight, aleatory uncertainties can affect the different flight phases at different levels (e.g., on the dynamics perturbations or stage combustion) and be combined to lead to a failure state of the space vehicle trajectory. After their propelled phase, the different stages reach successively their separation altitudes and may fall back into the ocean (see Fig. 3). Such a dynamic phase is of

Table 5
Results for the launch vehicle stage fallback test-case.

Approach	CMC[a]			FORM			SS[b]				
	$\widehat{m}_{P_f}$	$S^2_{\widehat{P}_f}$	RE	$\widehat{m}_{P_f}$	$S^2_{\widehat{P}_f}$	RE	RB	ν	$\widehat{m}_{P_f}$	$S^2_{\widehat{P}_f}$	RE	RB	ν
NRA[c]	1.18×10^{-4}	8.82×10^{-11}	7.97×10^{-2}	9.16×10^{-3}	1.11×10^{-4}	1.15	75.32	–	1.13×10^{-4}	4.98×10^{-9}	0.63	-6.09×10^{-2}	5.65×10^{-2}
ARA	1.27×10^{-4}	1.05×10^{-10}	8.07×10^{-2}	8.28×10^{-5}	–	–	−0.35	–	1.25×10^{-4}	1.52×10^{-9}	0.31	-1.57×10^{-2}	20.6

[a] NRA: $N_{\boldsymbol{\theta}} = 10^3$ samples, $N_{\mathbf{X}} = 10^3$ samples | ARA: $N_{\mathbf{X},\boldsymbol{\theta}} = 10^6$ samples.
[b] NRA: $N_{\boldsymbol{\theta}} = 10^3$ samples, $N_{\mathbf{X}} = 10^3$ samples/step | ARA: $N_{\mathbf{X},\boldsymbol{\theta}} = 10^3$ samples/step.
[c] Ref. (Gauss–Hermite with $N_{\boldsymbol{\theta}} = 1009$ samples + SS with $N_{\mathbf{X}} = 10^4$ samples/step): $\widehat{\widetilde{P}}_{\mathrm{f,ref}} = 1.20 \times 10^{-4}$.

Table 6
Results for the collision test-case.

Approach	CMC			FORM					SS				
	$\widehat{m}_{P_f}$	$S^2_{\widehat{P}_f}$	RE	$\widehat{m}_{P_f}$	$S^2_{\widehat{P}_f}$	RE	RB	ν	$\widehat{m}_{P_f}$	$S^2_{\widehat{P}_f}$	RE	RB	ν
NRA	untractable	–	–	untractable	–	–	–	–	untractable	–	–	–	–
ARA	3.55×10^{-4}[a]	4.33×10^{-9}	0.19	0.10	–	–	293	–	3.68×10^{-4}[b]	6.96×10^{-8}	0.72	3.74×10^{-2}	1.32

[a] ARA: $N_{\mathbf{X},\boldsymbol{\theta}} = 10^5$ samples.
[b] ARA: $N_{\mathbf{X},\boldsymbol{\theta}} = 2 \times 10^{-3}$ samples/step.

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Fig. 3. Illustration scheme of the first stage fallback phase into the Atlantic Ocean.

Table 7
Second-order statistics and distributions of input random variables for the nonlinear oscillator test-case.

Variable X_i[a]	Distribution	Mean μ_{X_i}	C_v δ_{X_i}[b]
$X_1 = m_p$ (kg)	Lognormal	1.5	10%
$X_2 = m_s$ (kg)	Lognormal	0.01	10%
$X_3 = k_p$ (N m^{-1})	Lognormal	1	20%
$X_4 = k_s$ (N m^{-1})	Lognormal	0.01	20%
$X_5 = \zeta_p$ (1)	Lognormal	0.05	40%
$X_6 = \zeta_s$ (1)	Lognormal	0.02	50%
$X_7 = F_s$ (N)	Lognormal	μ_{X_7} uncertain[c]	10%
$X_8 = S_0$ (m s^{-2})	Lognormal	100	10%
$\Theta = \mu_{X_7}$ (N)	Normal	21.5	10%

[a] The basic variables are independent.
[b] Note that the coefficient of variation (C_v) of X_i is defined by $\delta_{X_i} = \sigma_{X_i}/|\mu_{X_i}|$ for $\mu_{X_i} \neq 0$.
[c] For a fixed value $\mu_{X_7} = 21.5$, $p_{\text{f,ref}} = 4.75 \times 10^{-5}$.

utmost importance in terms of launcher safety since the consequence of a mistake in the prediction of the fallback zone can be dramatic in terms of human security and environmental impact. For that reason, the handling of uncertainties plays a crucial role in the comprehension and prediction of the global system behavior. That is the reason why it is of utmost importance to take it into account during the reliability analysis.

The black-box model $\mathcal{M}(\cdot)$ considered here is a trajectory simulation code of the dynamic fallback phase of a generic launcher first stage. For the interested reader, two different but close test-cases (with different numerical values) are used in [4] and in [53]. As an input vector of the simulation code, the following basic variables representing the initial conditions and the launch vehicle characteristics will be passed through the code:

X_1: stage altitude perturbation at separation (Δh (m));
X_2: velocity perturbation at separation (Δv (m s^{-1}));
X_3: flight path angle perturbation at separation ($\Delta\gamma$ (rad));
X_4: azimuth angle perturbation at separation ($\Delta\psi$ (rad));
X_5: propellant mass perturbation at separation (Δm (kg));
X_6: drag force error perturbation (ΔC_d dimensionless).

Moreover, in this case, the mean values of the basic variables X_2 and X_3 are considered as uncertain (see Table 8) as they are physical quantities difficult to measure and to control in reality. As an output, the code will give back the distance $D_{\text{code}} = \mathcal{M}(\mathbf{X})$, which is also a random variable, between the theoretical fallback position into the ocean and the estimated one due to the uncertainty propagation. The system failure is considered if the distance D_{code} exceeds a threshold safety distance d_{safe}:

$$g(\mathbf{X}) = \text{d}_{\text{safe}} - \mathcal{M}(\mathbf{X}) = \text{d}_{\text{safe}} - D_{\text{code}}. \tag{19}$$

In the numerical experiment, the threshold safety distance d_{safe} is chosen to be equal to 20 km so as to reach a reference probability without parameter uncertainty $p_{\text{f,ref}}$ equal to 2.31×10^{-7} (estimated by CMC with 10^8 samples and confirmed by SS with 10^3 samples/step).

Table 8
Second-order statistics and distributions of input random variables for the launch vehicle stage fallback test-case.

Variable X_i[a]	Distribution	Mean μ_{X_i}	Std σ_{X_i}
$X_1 = \Delta h$ (m)	Normal	0	1650
$X_2 = \Delta v$ (m s^{-1})	Normal	μ_{X_2} uncertain[b]	3.7
$X_3 = \Delta\gamma$ (rad)	Normal	μ_{X_3} uncertain	0.001
$X_4 = \Delta\psi$ (rad)	Normal	0	0.0018
$X_5 = \Delta m$ (kg)	Normal	0	70
$X_6 = \Delta C_d$ (1)	Normal	0	0.1
$\Theta_2 = \mu_{X_2}$ (m s^{-1})	Normal	0	3.7
$\Theta_3 = \mu_{X_3}$ (rad)	Normal	0	0.001

[a] The basic variables are independent.
[b] For fixed values $\mu_{X_2} = 0$ and $\mu_{X_3} = 0$, $p_{\text{f,ref}} = 2.31 \times 10^{-7}$.

Results. Numerical results gathered in Table 5 show that both NRA/CMC and ARA/CMC give similar results and manage to correctly estimate the predictive failure probability (whose reference value is given in Table 5). NRA/SS and ARA/SS even if a significant value of the efficiency ($\nu > 20$) for ARA/SS which indicates how promising is the use of ARA/SS with such an industrial test-case. As for NRA/FORM and ARA/FORM, they both give poor results. ARA/FORM gives at least an order of magnitude of the predictive failure probability quite close to the reference one.

Through this test-case, one can illustrate the budget allocation problem which appears in NRA. Classical reference failure probability without parameter uncertainty $p_{\text{f,ref}}$ is very low (order of magnitude of 10^{-7}, see Table 8) and can be time-consuming to get, especially with an expensive computer model. Adding a second integration loop over $\mathcal{D}_{\boldsymbol{\theta}}$ to get the reference predictive failure probability $\widehat{\widehat{P}}_{\text{f,ref}}$ can be untractable in a context of rare event. The problem concerning the simulation budget allocation (between the two domains, $\mathcal{D}_{\mathbf{X}}$ and $\mathcal{D}_{\boldsymbol{\Theta}}$) can be an obstacle to an accurate estimation of the predictive failure probability when using CMC-based

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methods. For instance, for a fixed simulation budget, one needs to decide whether to allow a substantial budget (i.e. more samples) to get a better precision over the integral on $\mathcal{D}_{\mathbf{X}}$ or over the integral on $\mathcal{D}_{\boldsymbol{\Theta}}$. On the one hand, because the simulation budget over $\mathcal{D}_{\mathbf{X}}$ is not easily reducible as it directly affects the estimation accuracy of the failure probability, it comes that adding the second integration budget over $\mathcal{D}_{\boldsymbol{\Theta}}$ can be computationally critical. On the other hand, sampling with only a few number of points over $\mathcal{D}_{\boldsymbol{\Theta}}$ may introduce a bias in the final measure of reliability by advantaging some parameter values which influence the final probability measure without taking their relative weight into account. For all these reasons, a *design of experiments* (DOE) has been conducted so as to optimize the sampling on $\mathcal{D}_{\boldsymbol{\Theta}}$. To do so, we decided to use a *quadrature scheme*-based DOE over $\mathcal{D}_{\boldsymbol{\Theta}}$. The idea is to approximate a k-variate integral over $\mathcal{D}_{\boldsymbol{\Theta}} \subseteq \mathbb{R}^k$ of the form:

$$\mathcal{I}[P_{\mathrm{f}}(\boldsymbol{\theta})] = \int_{\mathcal{D}_{\boldsymbol{\Theta}}} P_{\mathrm{f}}(\boldsymbol{\theta}) f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \tag{20}$$

where $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \equiv w(\boldsymbol{\theta})$ is a density (or weight) function which is evaluated on gridpoints. The quadrature rule provides an approximation using a combination of these weight functions such that [54]:

$$\begin{aligned}\mathcal{I}[P_{\mathrm{f}}(\boldsymbol{\theta})] &\approx \sum_{j_1=1}^{M_1} \sum_{j_2=1}^{M_2} \cdots \sum_{j_k=1}^{M_k} (w_{j_1} \otimes w_{j_2} \otimes \cdots \otimes w_{j_k}) \\ &\quad \times P_{\mathrm{f}}(\theta_1^{(j_1)}, \theta_2^{(j_2)}, \ldots, \theta_k^{(j_k)})\end{aligned} \tag{21}$$

with w_j the weights and $\otimes$ the tensor product operator. The indices $M_1, \ldots, M_k$ represents the number of points in each dimension. In the specific case of launch vehicle stage fallback, we have two uncertain distribution parameters, which means that the integration domain is $\mathbb{R}^2$. The quadrature type is chosen to be a *Gauss–Hermite quadrature scheme*, which means that we use Gaussian weights [55,56]. Depending on the problem dimensionality, one can choose an accuracy level M_{acc} which allows to integrate complete polynomials of total order $2M_{acc} - 1$ exactly. Here, we chose an accuracy level $M_{acc} = 14$ so as to provide enough samples (here, it corresponds exactly to 1009 samples [55]) to cover the domain $\mathcal{D}_{\boldsymbol{\Theta}}$. Such a choice is constrained by the expensive aspect of the computer code. However, for different applications, one could choose another accuracy level. Finally, coupling this DOE with a SS method with 10^4 samples/step allows to estimate the reference predictive failure probability $\widehat{\widehat{P}}_{\mathrm{f,ref}}$. A last remark concerns the fact that taking only two parameters out of six basic variables as being uncertain implies to increase the failure probability of three logarithmic decades in terms of magnitude compared to the classical reference estimate. Again, that emphasizes how crucial taking distribution parameters uncertainty is during the reliability analysis phase.

4.4. Application on estimation of collision probability between orbiting objects

Description. Because of the drastic growth of the number of orbiting objects (cataloged and uncataloged space debris) over the past few decades, the number of potential collision between satellites and other orbiting objects increased (see Fig. 4). Space debris surveillance and management is one of the key issue and is directly linked to the rare event probability estimation topic.

In this test-case, the failure scenario concerns a collision between a space debris and a satellite, both orbiting around an Earth-centered inertial reference frame. The dynamical model used here is a deterministic model $\mathcal{M}(\cdot)$ computing the minimum distance $D_{\min}$ between the debris and the satellite during a given time span τ. For the interested reader, two other applications (with different numerical values) of a similar test-case are treated in [4] and in [34] and another approach to estimate the probability of collision can be found in [57]. Assuming that the position and the speed of the satellite are perfectly known, the input vector of basic variables gathers the three components of the space debris position vector and the three components of its speed vector [58]. Here, the mean values of the space debris position vector are uncertain (see Table 9) because of the difficulty to measure accurately these quantities. As for the correlation structure of the basic variables, it is given by the following linear correlation matrix $\mathbf{R} = [\rho_{ij}]_{i,j \in \{1,\ldots,d\}}$:

$$\mathbf{R} = \begin{bmatrix} 1 & 0.97 & 0.12 & 0.84 & -0.85 & -0.98 \\ & 1 & 0.10 & 0.73 & 0 & 0 \\ & & 1 & -0.14 & -0.47 & -0.11 \\ & & & 1 & -0.79 & -0.81 \\ & (\mathrm{sym.}) & & & 1 & 0.83 \\ & & & & & 1 \end{bmatrix}. \tag{22}$$

Thus the lsf can be expressed as follows:

$$g(\mathbf{X}) = \mathrm{d}_{\mathrm{collision}} - \mathcal{M}(\mathbf{X}) = \mathrm{d}_{\mathrm{collision}} - D_{\min}. \tag{23}$$

In the numerical experiment, the threshold collision distance $\mathrm{d}_{\mathrm{collision}}$ is chosen to be equal to 20 m so as to reach a reference probability without parameter uncertainty $p_{\mathrm{f,ref}}$ equal to 3.4×10^{-4} (estimated by CMC with 10^6 samples and confirmed with a SS with 10^4 samples/step).

Results. This industrial test-case can be considered as the worst case here since it involves six correlated basic random variables with three uncertain distribution parameters. Numerical results are given in Table 6. Again, since we know the conditional marginal



Fig. 4. Picture extracted from a video (*The story of space debris*) by ESA© (http://www.esa.int).

Table 9
Second-order statistics and distributions of input random variables for the collision test-case.

Variable X_i [a]	Distribution	Mean μ_{X_i}	Std σ_{X_i}
$X_1 = r_{deb,1}$ (km)	Normal	μ_{X_1} uncertain[b]	10%
$X_2 = r_{deb,2}$ (km)	Normal	μ_{X_2} uncertain	10%
$X_3 = r_{deb,3}$ (km)	Normal	μ_{X_3} uncertain	20%
$X_4 = v_{deb,1}$ (km s^{-1})	Normal	0	20%
$X_5 = v_{deb,2}$ (km s^{-1})	Normal	0	40%
$X_6 = v_{deb,3}$ (km s^{-1})	Normal	0	50%
$\Theta_1 = \mu_{X_1}$ (km)	Normal	0	0.0625
$\Theta_2 = \mu_{X_2}$ (km)	Normal	0	0.04
$\Theta_3 = \mu_{X_3}$ (km)	Normal	0	0.000625

[a] Linear correlation structure given by the matrix $\mathbf{R}$ in Eq. (22).
[b] For fixed values $\mu_{X_1} = \mu_{X_2} = \mu_{X_3} = 0$, $p_{\mathrm{f,ref}} = 3.4 \times 10^{-4}$.

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pdfs and the covariance matrix, we can apply the same approach using the adapted Rosenblatt transformation. Here, none of the nested approaches are accessible computationally speaking since the model is very expensive to evaluate. This is worse if we want to get a reference result using a huge CMC. Moreover, the strategy using quadrature-based scheme and subset simulations as used in subsection 4.3 is not conceivable here since it would require to truncate the scheme at a very high order so as to get enough points over $\mathcal{D}_{\boldsymbol{\Theta}}$ to get relevant results. For all these reasons, ARA seems to be the only alternative to assess reliability under distribution parameter uncertainty regarding a realistic simulation budget constraint. ARA/CMC gives a result confirmed by ARA/SS with 10^3 samples/step. However, the ν factor is just above one and does not indicate a huge efficiency compared to ARA/CMC. However, ARA/SS offers still more tuning possibilities in the algorithm and can handle lower probabilities. ARA/FORM definitely fails to assess a correct probability. As a consequence, ARA/SS remains the only solution to possibly assess reliability considering uncertain distribution parameters. Even if the efficiency is not as good as the one obtained for previous test-cases, it still highlights the fact that this method allows benefits compared to ARA/CMC.

4.5. Synthesis about aerospace test-cases

The aim of this subsection is to give a synthesis for the interested reader to get the main advantages and drawbacks of both NRA and ARA. According to the numerical results, one can sum up the following characteristics:

- ARA leads to more accurate results than NRA with respect to a given simulation budget;
- only ARA is able to handle very expensive simulation codes;
- NRA suffers from the "budget allocation" problem;
- ARA requires to adapt the Rosenblatt transformation so as to use the classical reliability methods in the standard normal space.

As a final remark concerning the coupling between ARA and reliability methods, one can notice that ARA/SS seems to be the most generic method since it can handle most of the difficulties encountered in complex simulation codes. However, other methods can be used if some specific characteristics are preponderant. For instance, a linear limit-state function should lead to use ARA/FORM instead (regardless the rareness of the failure probability). Combining several characteristics lead to deduce which optimal method should be used regarding all these simulation constraints.

5. Limitations and possible enhancements

As described in previous sections, reliability assessment under parameter uncertainty involves mainly two components: the choice of an estimator for the failure probability (here, the predictive failure probability $\widetilde{P}_{\mathrm{f}}$) and the choice of a numerical strategy to get an estimate of it (i.e. NRA vs. ARA). Concerning the first point, $\widetilde{P}_{\mathrm{f}}$ corresponds to the mean failure probability over all the conditional failure probabilities $P_{\mathrm{f}}(\boldsymbol{\theta})$. As discussed in [38, 39], such a predictive estimator appears to be, from the *statistical decision theory* [59] point of view, associated to a *quadratic cost function* whose aim is to quantify the impact of a mis-estimation through $\widetilde{P}_{\mathrm{f}}$ with respect to the true failure probability p_{f} (probability obtained with a full knowledge of the probability distribution of $\mathbf{X}$ and with a perfect computer model). However, as pointed out in [39], since the associated quadratic cost function is symmetric, both under- and over-estimations have the same costs, theoretically speaking. Nevertheless, this symmetry of costs may have asymmetric consequences from a risk management perspective. Thus, a more conservative estimate (e.g., a quantile) could be combined with $\widetilde{P}_{\mathrm{f}}$ to more accurately characterize the probability of interest. However, estimating such an indicator is possible with NRA but, as mentioned previously, with a computational cost incompatible with industrial considerations. Further investigations are required to make the calculation of quantiles possible with ARA. In order to reduce the computational cost of these estimations, one possible enhancement track could be to use a *surrogate model* [4]. However, the use of surrogate models also raises several difficulties. Firstly, the construction of such a model over the whole definition domain (required for quantile estimation mentioned previously) of the input random variables can be a challenging task. Secondly, the performance of a surrogate mainly depends on the tuning of hyper-parameters. Thirdly, the surrogate model induces its own uncertainty that has to be handled.

6. Conclusions

In this article, we compared two different approaches to handle reliability assessment under distribution parameter uncertainty. If the first one, the nested reliability approach (NRA) is widely used and simple to set up, it definitely crashes with both the curse of dimensionality and simulation budget considerations. The second one, the augmented reliability approach (ARA) relies on the definition of an augmented input vector of uncertain distribution parameters and the basic input variables. For ARA, numerical sampling and integration can be carried simultaneously on both basic random variables conditioned on uncertain distribution parameters. The main principles of both methods have been presented into a unified common framework. Specific attention has been given to the algorithmic links and differences existing between these approaches. Specificities concerning the use of Rosenblatt transformation with ARA have been evoked. Then, a comparison between NRA and ARA has been carried out through application on both academic and industrial test-cases (launch vehicle stage fallback zone estimation and collision probability estimation between orbiting object and space debris), the final ones being representative of the complex simulation codes used in aerospace engineering. This study showed the benefits of using ARA with dedicated rare event probability estimation methods such as Subset Simulations for complex models with nonlinear lsfs and correlated inputs. Several enhancements raised in Section 5 are current open research tracks.

Conflict of interest statement

None declared.

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