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Geometric obstructions for Fredholm boundary conditions for manifolds with corners*

P. Carrillo Rouse and J.M. Lescure

March 16, 2017

Abstract

For every connected manifold with corners we introduce a very computable homology theory called conormal homology, defined in terms of faces and incidences and whose cycles correspond geometrically to corner's cycles. Its Euler characteristic (over the rationals, dimension of the total even space minus the dimension of the total odd space), $\chi_{\mathrm{cn}} := \chi_0 - \chi_1$, is given by the alternated sum of the number of (open) faces of a given codimension.
The main result of the present paper is that for a compact connected manifold with corners X given as a finite product of manifolds with corners of codimension less or equal to three we have that
1) If X satisfies the Fredholm Perturbation property (every elliptic pseudodifferential b-operator on X can be perturbed by a b-regularizing operator so it becomes Fredholm) then the even Euler corner character of X vanishes, *i.e.* $\chi_0(X) = 0$.
2) If the even Periodic conormal homology group vanishes, *i.e.* $H_0^{\mathrm{pcn}}(X) = 0$ then X satisfies the stably homotopic Fredholm Perturbation property (i.e. every elliptic pseudodifferential b-operator on X satisfies the same named property up to stable homotopy among elliptic operators).
3) If $H_0^{\mathrm{pcn}}(X)$ is torsion free and if the even Euler corner character of X vanishes, *i.e.* $\chi_0(X) = 0$ then X satisfies the stably homotopic Fredholm Perturbation property. For example for every finite product of manifolds with corners of codimension at most two the conormal homology groups are torsion free.
The main theorem behind the above result is the explicit computation in terms of conormal homology of the $K-$theory groups of the algebra $\mathcal{K}_b(X)$ of b-compact operators for X as above. Our computation unifies the only general cases covered before, for codimension zero (smooth manifolds) and for codimension 1 (smooth manifolds with boundary).

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1 Introduction

In a smooth compact manifold the vanishing of the Fredholm (Analytic) index of an elliptic (=Fredholm in this case) pseudodifferential operator is equivalent to the invertibility, up to perturabtion by a regularizing operator, of the operator. In the case of a smooth manifold with smooth boundary, not every elliptic (b-operator) totally characteristic pseudodifferential operator is Fredholm but it can be endowed with Fredholm boundary conditions, that is it can be perturbed with a regularizing operator to become Fredholm. This non trivial fact, which goes back to Atiyah, Patodi and Singer [2], can also be obtained from the vanishing of a boundary analytic index (see [19] or below for more details). In fact, in this case the boundary analytic index takes values in the K_0-theory group of the algebra of regularizing operators and this K-theory group is easily seen to vanish. Now, for manifolds with corners from codimension 2 (for example a square) nothing was known in general. In this paper we will show that the global topology/geometry of the corners and the way the corners form cycles enter in a fundamental way in a primary obstruction to give Fredholm boundary conditions. As we will see the answer passes by the computation of some K-theory groups. We explain now with more details the problem and the content of this paper.

Using K-theoretical tools for solving index problems was the main asset in the series of papers by Atiyah-Singer ([3, 4]) in which they introduce and prove several index formulas for smooth compact manifolds. For the case of a manifold with boundary, Atiyah-Patodi-Singer used different tools in [2] to give a formula for the Fredholm index of a Dirac type operator with the so called APS boundary condition. It is without mentioning the importance of these results in modern mathematics. Still, besides several very interesting examples (mainly of codimension 2) and higher/more general versions of the two cases above, almost nothing is known in general for manifolds with corners or for other kind of manifolds with singularities, neither explicit topological formulas nor general geometric properties to ensure Fredholmness. Putting an appropriate K-theory setting has been part of the problem for several years.

In [16], Melrose and Piazza construct an algebra of pseudodifferential operators $\Psi_b^*(X)$ associated to any manifold with corners[1] X. The elements in this algebra are called $b-$pseudodifferential operators[2], the subscript b identifies these operators as obtained by "microlocalization" of the Lie algebra of C^∞ vector fields on X tangent to the boundary. This Lie algebra of vector fields can be explicitly obtained as sections of the so called b-tangent bunlde ${}^bTX$ (compressed tangent bundle that we will recall below). The b-pseudodifferential calculus developed by Melrose and Piazza has the classic and expected properties. In particular there is a principal symbol map

$$\sigma_b : \Psi_b^m(X) \to S^{[m]}({}^bT^*X).$$

Ellipticity has the usual meaning, namely invertibility of the principal symbol. Moreover (discussion below theorem 2.15 in ref.cit.), an operator is elliptic if and only[3] if it has an quasi-inverse modulo $\Psi_b^{-\infty}(X)$. Now, the operators in $\Psi_b^{-\infty}(X)$ are not all compact (unless the topological boundary $\partial X = \emptyset$) but they contain a subalgebra consisting of compact operators (those for which certain indicial map is zero, p.8 ref.cit.). Hence, among elliptic b-pseudodifferential operators one has those for which the quasi-inverse is actually modulo compact operators and hence Fredholm[4]. These b-elliptic operators are called fully elliptic operators for reasons explained below.

Now, as every 0-order b-pseudodifferential operator, the operators in $\Psi_b^{-\infty}(X)$ extend to bounded operators on $L^2(X)$ and hence if we consider its completion as bounded operators one obtains an

[1] In this paper we will always assume X to be connected

[2] To simplify we discuss only the case of scalar opertaros, the passage to operators acting on sections of vector bundles is done in the classic way.

[3] Notice that this remark implies that to a b-pseudodifferential operator one can associate an "index" in the algebraic K-theory group $K_0(\Psi_b^{-\infty}(X))$ (classic construction of quasi-inverses).

[4] see p.8 ref.cit. for a characterization of Fredholm operators in terms of an indicial map or [14] thm 2.3 for the proof of Fully ellipticity iff Fredholm

algebra denoted in this paper by $\mathcal{K}_b(X)$ that fits in a short exact sequence of C^*-algebras of the form

$$0 \longrightarrow \mathcal{K}(X) \xrightarrow{i_0} \mathcal{K}_b(X) \xrightarrow{r} \mathcal{K}_b(\partial X) \longrightarrow 0 \tag{1.1}$$

where $\mathcal{K}(X)$ is the algebra of compact operators in $L^2(X)$. In order to study Fredholm problems and analytic index problems one has to understand the K-theory of the above short exact sequence.

To better explain how these K-theory groups enter into the study of Fredholm Perturbation properties and in order to enounce our first main results we need to settle some definitions.

Analytic and Boundary analytic Index morphism: Given an elliptic b-pseudodifferential D, the classic construction of parametrices adapts to give a $K-$theory valued index in $K_0(\mathcal{K}_b(X))$ that only depends on its principal symbol class $\sigma_b(D) \in K^0_{top}({}^bT^*X)$. In more precise terms, the short exact sequence

$$0 \longrightarrow \mathcal{K}_b(X) \longrightarrow \overline{\Psi^0_b(X)} \xrightarrow{\sigma_b} C({}^bS^*X) \longrightarrow 0 \tag{1.2}$$

gives rise to $K-$theory index morphism $K_1(C({}^bS^*X)) \to K_0(\mathcal{K}_b(X))$ that factors in a canonical way by an index morphism

$$K^0_{top}({}^bT^*X) \xrightarrow{Ind^a_X} K_0(\mathcal{K}_b(X)) \tag{1.3}$$

called *the Analytic Index morphism of* X. By composing the Analytic index with morphism induced by the restriction to the boundary we have a morphism

$$K^0_{top}({}^bT^*X) \xrightarrow{Ind^\partial_X} K_0(\mathcal{K}_b(\partial X)) \tag{1.4}$$

called *the Boundary analytic index morphism of* X. In fact $r : K_0(\mathcal{K}_b(X)) \to K_0(\mathcal{K}_b(\partial X)$ is an isomorphism from codimension greater or equal to one, proposition 5.6, and so the two indices above are the same modulo this isomorphism. In other words we completely understand the six term short exact sequence in K-theory associated to the sequence (1.1). Notice that in particular there is no contribution of the Fredholm index in the K_0-analytic index.

To state the next theorem we need to define the Fredholm Perturbation Property and its stably homotopic version.

Definition 1.1 *Let $D \in \Psi^m_b(X)$ be elliptic. We say that D satisfies:*

- *the Fredholm Perturbation Property ($\mathcal{FP}$) if there is $R \in \Psi_b^{-\infty}(X)$ such that $D + R$ is fully elliptic.*
- *the stably homotopic Fredholm Perturbation Property ($\mathcal{HFP}$) if there is a fully elliptic operator D' with $[\sigma_b(D')] = [\sigma_b(D)] \in K_0(C^*({}^bTX))$.*

We also say that X satisfies the *(stably homotopic) Fredholm Perturbation Property* if any elliptic b-operator on X satisfies ($\mathcal{HFP}$).

Property ($\mathcal{FP}$) is of course stronger than property ($\mathcal{HFP}$). In [22], Nistor characterized ($\mathcal{FP}$) in terms of the vanishing of an index in some particular cases. In [21], Nazaikinskii, Savin and Sternin characterized ($\mathcal{HFP}$) for arbitrary manifolds with corners using an index map associated with their dual manifold construction. We now rephrase the result of [21] and we give a new proof in terms of deformation groupoids.

Theorem 1.2 *Let D be an elliptic b-pseudodifferential operator on a compact manifold with corners X. Then D satisfies ($\mathcal{HFP}$) if and only if $Ind_\partial([\sigma_b(D)]) = 0$.*
In particular if D satisfies ($\mathcal{FP}$) then its boundary analytic index vanishes.

The above results fit exactly with the K-theory vs Index theory Atiyah-Singer's program and in that sense it is not completely unexpected. Now, in order to give a full characterization of the Fredholm perturbation property one is first led to compute or understand the K-theory groups for the algebras (1.1) preferably in terms of the geometry/topology of the manifold with corners. As it happens, the only general computations known previous to the present paper are:

- the K-theory of the compact operators $\mathcal{K}(X)$, giving $K_0(\mathcal{K}(X)) = \mathbf{Z}$ and $K_1(\mathcal{K}(X)) = 0$, which is of course essential for classic index theory purposes; and

- the K-theory of $\mathcal{K}_b(X)$ for a smooth manifold with boundary, giving $K_0(\mathcal{K}_b(X)) = 0$ and $K_1(\mathcal{K}_b(X)) = \mathbf{Z}^{1-p}$ with p the number of boundary components, which has the non trivial consequence that any elliptic b-operator on a manifold with boundary can be endowed with Fredholm boundary conditions.

Computation of the K-theory groups in terms of corner cycles. In this paper we explicitly compute the above K-theory groups for any finite product of manifolds with corners of codimension ≤ 3 in terms corner cycles (explanation below). Our computations and results are based on a geometric interpretation of the algebras of b-pseudodifferential operators in terms of Lie groupoids. We explain and recall the basic facts on groupoids and the b-pseudodifferential calculus in the first two sections. Besides being extremely useful to compute $K-$theory groups, the groupoid approach we propose reveals to be very powerful to compute index morphisms and relate several indices. Indeed, the relation between the different indices for manifolds with corners was only partially understood for some examples. Let us explain this in detail. Let X be a manifold with corners. Let $F_p = F_p(X)$ be the set of (without boundary, connected) faces of X of codimension p. To compute $K_*(\mathcal{K}_b(X))$, we use an increasing filtration of X given by the open subspaces:

$$X_p = \bigcup_{k\leq p \ ; \ f\in F_k} f. \tag{1.5}$$

We have $X_0 = \overset{\circ}{X}$ and $X_d = X$. We extend if necessary the filtration over $\mathbb{Z}$ by setting $X_k = \emptyset$ if $k < 0$ and $X_k = X$ if $k > d$. The C^*-algebra of $\mathcal{K}_b(X)$ inherits (for entire details see section 5) an increasing filtration by C^*-ideals:

$$\mathcal{K}(L^2(\overset{\circ}{X})) = A_0 \subset A_1 \subset \dots \ A_d = A = \mathcal{K}_b(X). \tag{1.6}$$

Using this filtration one can expect to compute the K-theory groups by computing the associated spectral sequence. In particular this means computing the differentials of this spectral sequence which is in general a very hard task. In fact, a similar filtration to the one above was considered by Melrose and Nistor in [15] and their main theorem is the computation of the first differential (theorem 9 ref.cit.). In trying to compute the higher differentials we were led to introduce a very computable (as shown below) homology theory for any manifold with corners computing the K-theory groups.

The idea to introduce what we call the conormal homology groups is very simple, and as we will justify, this homology theory is very easy to compute, for full details see section 4. With the notations above consider the following maps of free $\mathbb{Z}$-module generated by the faces:

$$\delta_p : \mathbb{Z}^{\#F_p} \to \mathbb{Z}^{\#F_{p-1}} \tag{1.7}$$

given by

$$\delta_p(f) = \sum_{\substack{g\in F_{p-1)} \\ f\subset \overline{g}}} (-1)^{\sigma_{f,g}} \cdot g \tag{1.8}$$

where f is a codimension p-face, and for every $p-1$ codimension face with $f \subset \overline{g}$ we define $\sigma_{f,g}$ as $+1$ or -1 depending one the position of f with respect to g. For be completely precise instead of signs $+1$ or -1 we were led to use volume elements of the normal bundles of the faces, in this way we obtained maps as above with good homological properties using exterior product and contractions (again see section 4 for entire details). To resume, consider

$$\mathcal{C}_0^{\mathrm{pcn}}(X) := \bigoplus_{n\geq 0} \mathbb{Z}^{\#F_{2n}} \text{ and } \mathcal{C}_1^{\mathrm{pcn}}(X) := \bigoplus_{n\geq 0} \mathbb{Z}^{\#F_{2n+1}} \tag{1.9}$$

and the following two homorphisms

$$\mathcal{C}_0^{\mathrm{pcn}}(X) \xrightarrow{\delta} \mathcal{C}_1^{\mathrm{pcn}}(X) \tag{1.10}$$

and

$$\mathcal{C}_1^{\mathrm{pcn}}(X) \xrightarrow{\delta} \mathcal{C}_0^{\mathrm{pcn}}(X) \tag{1.11}$$

naturally induced from the δ_p above. In proposition 7.1 we show that $\delta \circ \delta = 0$,from which we introduce

Definition 1.3 (Periodic conormal homology) *We call the periodic conormal homology groups of X and we denote by $H_i^{\mathrm{pcn}}(X)$, for $i = 0, 1$, the homology groups (even or odd respectively) defined by the total differential δ.*

Our main K-theory computation can be now be stated (theorem 5.8):

Theorem 1.4 *Let $X = \Pi_i X_i$ be a finite product of manifolds with corners of codimension less or equal to three. There are natural isomorphisms*

$$H_0^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow[\cong]{\phi_X} K_0(\mathcal{K}_b(X))) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \textit{and} \qquad H_1^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow[\cong]{\phi_X} K_1(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{1.12}$$

In the case X contains a factor of codimension at most two or X is of codimension three the result holds even without tensoring by $\mathbb{Q}$.

We insist on the computability of the Periodic conormal homology that comes from the fact of the explicit description of the differentials δ_p above in terms of the faces and of its incidences. To continue let us introduce the Corner characters.

Definition 1.5 (Corner characters) *Let X be a manifold with corners. We define the even conormal character of X as the finite sum*

$$\chi_0(X) = dim_{\mathbb{Q}} H_0^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{1.13}$$

Similarly, we define the odd conormal character of X as the finite sum

$$\chi_1(X) = dim_{\mathbb{Q}} H_1^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{1.14}$$

We can consider as well

$$\chi(X) = \chi_0(X) - \chi_1(X), \tag{1.15}$$

then by definition

$$\chi(X) = 1 - \#F_1 + \#F_2 - \cdots + (-1)^d \#F_d \tag{1.16}$$

We refer to the integer $\chi(X)$ as the Euler conormal character of X.

In particular one can rewrite the theorem above to have, for X as in the statement,

$$K_0(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{\chi_0(X)} \tag{1.17}$$

$$K_1(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{\chi_1(X)}$$

and, in terms of the corner character,

$$\chi(X) = rank(K_0(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q}) - rank(K_1(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q}). \tag{1.18}$$

Or in the case X is a finite product of manifolds with corners of codimension at most 2 we even have

$$K_0(\mathcal{K}_b(X)) \simeq \mathbb{Z}^{\chi_0(X)} \qquad \text{and} \qquad K_1(\mathcal{K}_b(X)) \simeq \mathbb{Z}^{\chi_1(X)} \tag{1.19}$$

and also $\chi_{\mathrm{cn}}(X) = rank(K_0(\mathcal{K}_b(X))) - rank(K_1(\mathcal{K}_b(X)))$.

We can finally state the following primary obstruction Fredholm Perturbation theorem (theorem 6.2) in terms of corner's characters and corner's cycles.

Theorem 1.6 *Let X be a compact manifold with corners of codimension greater or equal to one. If X is a finite product of manifolds with corners of codimension less or equal to three we have that*

1. *If X satisfies the Fredholm Perturbation property then the even Euler corner character of X vanishes, i.e. $\chi_0(X) = 0$.*
2. *If the even Periodic conormal homology group vanishes, i.e. $H_0^{\mathrm{pcn}}(X) = 0$ then X satisfies the stably homotopic Fredholm Perturbation property.*
3. *If $H_0^{\mathrm{pcn}}(X)$ is torsion free and if the even Euler corner character of X vanishes, i.e. $\chi_0(X) = 0$ then X satisfies the stably homotopic Fredholm Perturbation property.*

We beleive that the results above hold beyond the case of finite products of manifolds with corners of codimension ≤ 3. On one side conormal homology can be defined and computed in all generality and in all examples we have the isomorphisms above still hold. The problem in general is to compute beyond the third differential of the associated spectral sequence for the K-theory groups for manifolds with corners of codimension greater or equal to four. This is technically a very hard task and besides explicit interesting examples become rare (not products). Another problem is related with the possible torsion of the conormal homology groups, indeed we prove in theorem 4.6 that for a finite product of manifolds with corners of codimension at most two these groups are torsion free and that the odd group for a three codimensional manifold wiht corner is torsion free as well. We think that in general these groups are torsion free but the combinatorics become very hard and one needs a good way to deal with all these data. We will discuss all these topics elsewhere.

Partial results in the direction of this paper were enterprised by several authors, we have already mentioned the seminal works of Melrose and Nistor in [15] and of Nazaikinskii, Savin and Sternin in [20] and [21]. In particular Melrose and Nistor start the computation of the $K-$groups of the algebra of zero order b-operators and some particular cases of Boundary analytic index morphisms as defined here (together with some topological formulas for them). Also, Nistor solves in [22] the Fredholm Perturbation problem for manifolds with corners of the form a canonical simplex times a smooth manifold. Let us mention also the work of Monthubert and Nistor, [19], in which they construct a classifying space associated to a manifold with corners whose $K-$theory can be in principle used to compute the analytic index above. We were very much inspired by all these works.

The theorems above show the importance and interest of compute the Analytic and the Fredholm indices associated to a manifold with corners and if possible in a unified way. Using K-theory as above, for the case of a smooth compact connected manifold, the computation we are mentioning is nothing else that the Atiyah-Singer index theorem, [3]. As we mentioned already, for manifolds with boundary, Atiyah-Patodi-Singer gave a formula for the Fredholm index of a Dirac type operator. In fact, with the groupoid approach to index theory, several authors have contributed to the now realizable idea that one can actually use these tools to have a nice $K-$theoretical framework and to actually compute more general index theorems as in the classic smooth case. For example, in our common work with Monthubert, [6], we give a topological formula for the Fredholm Index morphism for manifolds with boundary that will allow us in a sequel paper to compare with the APS formula and obtain geometric information on the eta invariant. In the second paper of this series we will generalize our results of [6] for general manifolds with corners by giving explicit topological index computations for the indices appearing above.

2 Melrose-Piazza b-calculus for manifolds with corners via groupoids

2.1 Preliminaries on groupoids, K-theory C^*-algebras and Pseudodifferential Calculus

All the material in this section is well known and by now classic for the people working in groupoid's C^*-algebras, K-theory and index theory. For more details and references the reader is sent to [8], [23], [18], [13], [10], [25], [1].

Groupoids: Let us start with the definition.

Definition 2.1 *A groupoid consists of the following data: two sets $\mathscr{G}$ and $\mathscr{G}^{(0)}$, and maps*

(1) $s, r : \mathscr{G} \to \mathscr{G}^{(0)}$ called the source and range (or target) map respectively,

(2) $m : \mathscr{G}^{(2)} \to \mathscr{G}$ called the product map (where $\mathscr{G}^{(2)} = \{(\gamma, \eta) \in \mathscr{G} \times \mathscr{G} : s(\gamma) = r(\eta)\}$),

such that there exist two maps, $u : \mathscr{G}^{(0)} \to \mathscr{G}$ (the unit map) and $i : \mathscr{G} \to \mathscr{G}$ (the inverse map), which, if we denote $m(\gamma, \eta) = \gamma \cdot \eta$, $u(x) = x$ and $i(\gamma) = \gamma^{-1}$, satisfy the following properties:

(i). $r(\gamma \cdot \eta) = r(\gamma)$ and $s(\gamma \cdot \eta) = s(\eta)$.

(ii). $\gamma \cdot (\eta \cdot \delta) = (\gamma \cdot \eta) \cdot \delta$, $\forall \gamma, \eta, \delta \in \mathscr{G}$ when this is possible.

(iii). $\gamma \cdot x = \gamma$ and $x \cdot \eta = \eta$, $\forall \gamma, \eta \in \mathscr{G}$ with $s(\gamma) = x$ and $r(\eta) = x$.

(iv). $\gamma \cdot \gamma^{-1} = u(r(\gamma))$ *and* $\gamma^{-1} \cdot \gamma = u(s(\gamma))$, $\forall \gamma \in \mathscr{G}$.

Generally, we denote a groupoid by $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$. *A morphism* f *from a groupoid* $\mathscr{H} \rightrightarrows \mathscr{H}^{(0)}$ *to a groupoid* $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ *is given by a map* f *from* $\mathscr{G}$ *to* $\mathscr{H}$ *which preserves the groupoid structure, i.e.* f *commutes with the source, target, unit, inverse maps, and respects the groupoid product in the sense that* $f(h_1 \cdot h_2) = f(h_1) \cdot f(h_2)$ *for any* $(h_1, h_2) \in \mathscr{H}^{(2)}$.

For A, B subsets of $\mathscr{G}^{(0)}$ we use the notation $\mathscr{G}_A^B$ for the subset

$$\{\gamma \in \mathscr{G} : s(\gamma) \in A,\, r(\gamma) \in B\}.$$

We will also need the following definition:

Definition 2.2 (Saturated subgroupoids) *Let* $\mathscr{G} \rightrightarrows M$ *be a groupoid.*

1. *A subset* $A \subset M$ *of the units is said to be saturated by* $\mathscr{G}$ *(or only saturated if the context is clear enough) if it is union of orbits of* $\mathscr{G}$.

2. *A subgroupoid*

$$\begin{array}{ccc} \mathscr{G}_1 & \subset & \mathscr{G} \\ {\scriptstyle r}\downarrow\downarrow{\scriptstyle s} & & {\scriptstyle r}\downarrow\downarrow{\scriptstyle s} \\ M_1 & \subset & M \end{array} \tag{2.1}$$

   is a saturated subgroupoid if its set of units $M_1 \subset M$ *is saturated by* $\mathscr{G}$.

A groupoid can be endowed with a structure of topological space, or manifold, for instance. In the case when $\mathscr{G}$ and $\mathscr{G}^{(0)}$ are smooth manifolds, and s, r, m, u are smooth maps (with s and r submmersions), then $\mathscr{G}$ is called a Lie groupoid. In the case of manifolds with boundary, or with corners, this notion can be generalized to that of continuous families groupoids (see [24]) or as Lie groupoids if one considers the category of smooth manifolds with corners.

C^*-algebras: To any Lie groupoid $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ one has several C^*-algebra completions for the *-convolution algebra $C_c^\infty(\mathscr{G})$. Since in this paper all the groupoids considered are amenable we will be denoting by $C^*(\mathscr{G})$ the maximal and hence reduced C^*-algebra of $\mathscr{G}$. From now on, all the groupoids are then going to be assumed amenable.

In the sequel we will use the following two results which hold in the generality of locally compact groupoids equipped with Haar systems.

1. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two locally compact groupoids equipped with Haar systems. Then for locally compact groupoid $\mathscr{G}_1 \times \mathscr{G}_2$ we have

$$C^*(\mathscr{G}_1 \times \mathscr{G}_2) \cong C^*(\mathscr{G}_1) \otimes C^*(\mathscr{G}_2). \tag{2.2}$$

2. Let $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ a locally compact groupoid with Haar system μ. Let $U \subset \mathscr{G}^{(0)}$ be a saturated open subset, then $F := \mathscr{G}^{(0)} \setminus U$ is a closed saturated subset. The Haar system μ can be restricted to the restriction groupoids $\mathscr{G}_U := \mathscr{G}_U^U \rightrightarrows U$ and $\mathscr{G}_F := \mathscr{G}_F^F \rightrightarrows F$, and we have the following short exact sequence of C^*-algebras:

$$0 \longrightarrow C^*(\mathscr{G}_U) \xrightarrow{\ i\ } C^*(\mathscr{G}) \xrightarrow{\ r\ } C^*(\mathscr{G}_F) \longrightarrow 0 \tag{2.3}$$

   where $i : C_c(\mathscr{G}_U) \to C_c(\mathscr{G})$ is the extension of functions by zero and $r : C_c(\mathscr{G}) \to C_c(\mathscr{G}_F)$ is the restriction of functions.

K-theory: We will be considering the K-theory groups of the C^*-algebra of a groupoid, for space purposes we will be denoting these groups by

$$K^*(\mathscr{G}) := K_*(C^*(\mathscr{G})). \tag{2.4}$$

We will use the classic properties of the K-theory functor, mainly its homotopy invariance and the six term exact sequence associated to a short exact sequence. Whenever the groupoid in question is a space (unit's groupoid) X we will use the notation

$$K^*_{top}(X) := K_*(C_0(X)). \tag{2.5}$$

to remark that in this case this group is indeed isomorphic to the topological $K-$theory group.

ΨDO Calculus for groupoids. A pseudodifferential operator on a Lie groupoid (or more generally a continuous family groupoid) $\mathscr{G}$ is a family of peudodifferential operators on the fibers of $\mathscr{G}$ (which are smooth manifolds without boundary), the family being equivariant under the natural action of $\mathscr{G}$.

Compactly supported pseudodifferential operators form an algebra, denoted by $\Psi^{\infty}(\mathscr{G})$. The algebra of order 0 pseudodifferential operators can be completed into a C^*-algebra, $\overline{\Psi^0}(\mathscr{G})$. There exists a symbol map, σ, whose kernel is $C^*(\mathscr{G})$. This gives rise to the following exact sequence:

$$0 \to C^*(\mathscr{G}) \to \overline{\Psi^0}(\mathscr{G}) \to C_0(S^*(\mathscr{G})) \to 0$$

where $S^*(\mathscr{G})$ is the cosphere bundle of the Lie algebroid of $\mathscr{G}$.

In the general context of index theory on groupoids, there is an analytic index which can be defined in two ways. The first way, classical, is to consider the boundary map of the 6-terms exact sequence in K-theory induced by the short exact sequence above:

$$ind_a : K_1(C_0(S^*(\mathscr{G}))) \to K_0(C^*(\mathscr{G})).$$

Actually, an alternative is to define it through the tangent groupoid of Connes, which was originally defined for the groupoid of a smooth manifold and later extended to the case of continuous family groupoids ([18, 12]). The tangent groupoid of a Lie groupoid $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ is a Lie groupoid

$$\mathscr{G}^{tan} = A(\mathscr{G}) \bigsqcup \mathscr{G} \times (0,1] \rightrightarrows \mathscr{G}^{(0)} \times [0,1],$$

with smooth structure given by the deformation to the normal cone construction, see for example [5] for a survey of this construction related with the tangent groupoid construction.

Using the evaluation maps, one has two K-theory morphisms, $e_0 : K_0(C^*(\mathscr{G}^{tan})) \to K^0_{top}(A\mathscr{G})$ which is an isomorphism (since $K_*(C^*(\mathscr{G} \times (0,1])) = 0$), and $e_1 : K_*(C^0(\mathscr{G}^{tan})) \to K_0(C^*(\mathscr{G}))$. The analytic index can be defined as

$$ind_a = e_1 \circ e_0^{-1} : K^0_{top}(A^*\mathscr{G}) \to K_0(C^*(\mathscr{G})).$$

modulo the surjection $K_1(C_0(S^*(\mathscr{G})) \to K^0(A^*\mathscr{G})$.

See [18, 23, 17, 12, 28] for a detailed presentation of pseudodifferential calculus on groupoids.

2.2 Melrose-Piazza b-calculus for manifolds with corners via the b-groupoid

We start by defining the manifolds with corners we will be using in the entire paper.

A manifold with corners is a Hausdorff space covered by compatible coordinate charts with coordinate functions modeled in the spaces

$$\mathbf{R}^n_k := [0, +\infty)^k \times \mathbf{R}^{n-k}$$

for fixed n and possibly variable k.

Definition 2.3 *A manifold with embedded corners X is a Hausdorff topological space endowed with a subalgebra $C^{\infty}(X) \in C^0(X)$ satisfying the following conditions:*

1. *there is a smooth manifold $\tilde{X}$ and a map $\iota : X \to \tilde{X}$ such that*

$$\iota^*(C^{\infty}(\tilde{X})) = C^{\infty}(X),$$

2. *there is a finite family of functions $\rho_i \in C^{\infty}(\tilde{X})$, called the defining functions of the hyperfaces, such that*

$$\iota(X) = \bigcap_{i \in I} \{\rho_i \geq 0\}.$$

3. *for any $J \subset I$,*

$$d_x\rho_i(x) \text{ are linearly independent in } T^*_x\tilde{X} \text{ for all } x \in F_J := \textstyle\bigcap_{i\in J}\{\rho_i = 0\}.$$

Terminology: In this paper we will only be considering manifolds with embedded corners. We will refer to them simply as manifolds with corners. We will also assume our manifolds to be connected. More general manifold with corners deserve attention but as we will see in further papers it will be more simple to consider them as stratified pseudomanifolds and desingularize them as manifolds with embedded corners with an iterated fibration structure.

Given a compact manifold corners X, Melrose and Piazza constructed in [16] the algebra $\Psi_b^*(X)$ of b-pseudodifferential operators. The elements in this algebra are called $b-$pseudodifferential operators, the subscript b identifies these operators as obtained by "microlocalization" of the Lie algebra of C^∞ vector fields on X tangent to the boundary. This Lie algebra of vector fields can be explicitly obtained as sections of the so called b-tangent bunlde ${}^bTX$ (compressed tangent bundle that we will appear below as the Lie algebroid of an explicit Lie groupoid). The b-pseudodifferential calculus developed by Melrose and Piazza has the classic and expected properties. In particular there is a principal symbol map

$$\sigma_b : \Psi_b^m(X) \to S^{[m]}({}^bT^*X).$$

Ellipticity has the usual meaning, namely invertibility of the principal symbol. Moreover (discussion below theorem 2.15 in ref.cit.), an operator is elliptic if and only[5] if it has an quasi-inverse modulo $\Psi_b^{-\infty}(X)$. Now, the operators in $\Psi_b^{-\infty}(X)$ are not all compact (unless the topological boundary $\partial X = \emptyset$) but they contain a subalgebra consisting of compact operators (those for which certain indicial map is zero, p.8 ref.cit.). Hence, among elliptic b-pseudodifferential operators one has those for which the quasi-inverse is actually modulo compact operators and hence Fredholm (again, see p.8 ref.cit. for a characterization of Fredholm operators in terms of an indicial map), these b-elliptic operators are called fully elliptic operators.

Now, as every 0-order b-pseudodifferential operator (ref.cit. (2.16)), the operators in $\Psi_b^{-\infty}(X)$ extend to bounded operators on $L^2(X)$ and hence if we consider its completion as bounded operators one obtains an algebra denoted in this paper by $\mathcal{K}_b(X)$ that fits in a short exact sequence of C^*-algebras of the form

$$0 \longrightarrow \mathcal{K}(X) \xrightarrow{i_0} \mathcal{K}_b(X) \xrightarrow{r} \mathcal{K}_b(\partial X) \longrightarrow 0 \tag{2.6}$$

where $\mathcal{K}(X)$ is the algebra of compact operators in $L^2(X)$.

Let X be a compact manifold with embedded corners, so by definition we are assuming there is a smooth compact manifold (of the same dimension) $\tilde{X}$ with $X \subset \tilde{X}$ and $\rho_1, ..., \rho_n$ defining functions of the faces. In [17], Monthubert constructed a Lie groupoid (called Puff groupoid) associated to any decoupage $(\tilde{X}, (\rho_i))$, it has the following expression

$$G(\tilde{X}, (\rho_i)) = \{(x, y, \lambda_1, ..., \lambda_n) \in \tilde{X} \times \tilde{X} \times \mathbb{R}^n : \rho_i(x) = e^{\lambda_i}\rho_i(y)\}. \tag{2.7}$$

as a Lie subgroupoid of $\tilde{X} \times \tilde{X} \times \mathbb{R}^k$. The Puff groupoid is not s-connected, denote by $G_c(\tilde{X}, (\rho_i))$ its s-connected component.

Definition 2.4 (The b-groupoid) *The $b-$groupoid $\Gamma_b(X)$ of X is by definition the restriction to X of the s-connected Puff groupoid (2.7) considered above, that is*

$$\Gamma_b(X) := G_c(\tilde{X}, (\rho_i))|_X \rightrightarrows X \tag{2.8}$$

The $b-$groupoid was introduced by B. Monthubert in order to give a groupoid description for the Melrose's algebra of b-pseudodifferential operators. We summarize below the main properties we will be using about this groupoid:

Theorem 2.5 (Monthubert [17]) *Let X be a manifold with corners as above, we have that*

1. *$\Gamma_b(X)$ is a $C^{0,\infty}$-amenable groupoid.*
2. *It has Lie algebroid $A(\Gamma_b(X)) =^b TX$, the b-tangent bundle of Melrose.*
3. *Its C^*-algebra (reduced or maximal is the same since amenability) coincides with the algebra of b-compact operators. The canonical isomorphism*

$$C^*(\Gamma_b(X)) \cong \mathcal{K}_b(X) \tag{2.9}$$

is given as usual by the Schwartz Kernel theorem.

[5]Notice that this remark implies that to a b-pseudodifferential operator one can associate an "index" in the algebraic K-theory group $K_0(\Psi_b^{-\infty}(X))$ (classic construction of quasi-inverses).

4. *The pseudodifferential calculus of $\Gamma_b(X)$ coincides with compactly supported b-calculus of Melrose-Piazza.*

Remark 2.6 *To simplify, in the present paper, we only discuss the case of scalar operators. The case of operators acting on sections of vector bundles is treated as classically by considering bundles of homomorphisms.*

3 Boundary analytic and Fredholm Indices for manifolds with corners: relations and Fredholm Perturbation characterization

We will now introduce the several index morphisms we will be using, mainly the Analytic and the Fredholm index. In all this section, X denotes a compact and connected manifold with embedded corners.

3.1 Analytic and Boundary analytic Index morphisms

Any elliptic b-pseudodifferential D has an analytical index $\mathrm{Ind}_{\mathrm{an}}(D)$ given by

$$\mathrm{Ind}_{\mathrm{an}}(D) = I([\sigma_b(D)]_1) \in K_0(\mathcal{K}_b(X))$$

where I is the connecting homomorphism in K-theory of exact sequence

$$0 \longrightarrow \mathcal{K}_b(X) \longrightarrow \overline{\Psi_b^0(X)} \xrightarrow{\sigma_b} C({}^bS^*X) \longrightarrow 0. \tag{3.1}$$

and $[\sigma_b(D)]_1$ is the class in $K_1(C({}^bS^*X)$ of the principal symbol $\sigma_b(D)$ of D.

Alternatively, we can express $\mathrm{Ind}(D)$ using adiabatic deformation groupoid of $\Gamma_b(X)$ and the class in K_0 of the same symbol, namely:

$$[\sigma_b(D)] = \delta([\sigma_b(D)]_1) \in K_0(C_0({}^bT^*X)) \tag{3.2}$$

where δ is the connecting homomorphism of the exact sequence relating the vector and sphere bundles:

$$0 \longrightarrow C_0({}^bT^*X) \longrightarrow C_0({}^bB^*X) \longrightarrow C({}^bS^*X) \longrightarrow 0. \tag{3.3}$$

Indeed, consider the exact sequence

$$0 \longrightarrow C^*(\Gamma_b(X) \times (0,1]) \longrightarrow C^*(\Gamma_b^{tan}(X)) \xrightarrow{r_0} C^*({}^bTX) \cong C_0({}^bT^*X) \longrightarrow 0, \tag{3.4}$$

in which the ideal is K-contractible and set

$$Ind_X^a = r_1 \circ r_0^{-1} : K_{top}^0({}^bT^*X) \longrightarrow K_0(\mathcal{K}_b(X)) \tag{3.5}$$

where $r_1 : K_0(C^*(\Gamma_b^{tan}(X))) \to K_0(C^*(\Gamma_b(X)))$ is induced by the restriction morphism to $t = 1$. Applying a mapping cone argument to the exact sequence (3.1) gives a commutative diagram

$$\begin{array}{ccc} K_1(C({}^bS^*X)) & \xrightarrow{\quad I \quad} & K_0(\mathcal{K}_b(X)) \\ & {\scriptstyle\delta}\searrow \quad \nearrow{\scriptstyle Ind_X^a} & \\ & K_{top}^0({}^bT^*X) & \end{array} \tag{3.6}$$

Therefore we get, as announced:

$$\mathrm{Ind}_{\mathrm{an}}(D) = Ind_X^a([\sigma_b(D)]) \tag{3.7}$$

The map Ind_X^a will be called the *Analytic Index morphism* of X. A closely related homomorphism is the *Boundary analytic Index morphism*, in which the restriction to $X \times \{1\}$ is replaced by the one to $\partial X \times \{1\}$, that is, we set:

$$Ind_X^\partial = r_\partial \circ r_0^{-1} : K_0(C_0({}^bT^*X) \longrightarrow K_0(C^*(\Gamma_b(X)|_{\partial X})), \tag{3.8}$$

where r_∂ is induced by the homomorphism $C^*(\Gamma_b^{tan}(X)) \longrightarrow C^*(\Gamma_b(X))|_{\partial X}$. We have of course

$$Ind_X^\partial = r_{1,\partial} \circ Ind_X^a \tag{3.9}$$

if $r_{1,\partial}$ denotes the map induced by the homomorphism $C^*(\Gamma_b(X)) \longrightarrow C^*(\Gamma_b(X)|_{\partial X})$.

3.2 Fredholm Index morphism

In general, elliptic b-operators on X are not Fredholm. Indeed, to construct an inverse of a b-operator modulo compact terms, we have to invert not only the principal symbol but also all the family of boundary symbols. One way to summarize this situation is to introduce the algebra of full, or joint, symbols. Let $\mathcal{H}$ be the set of closed boundary hyperfaces of X, and set

$$\mathcal{A}_{\mathcal{F}} = \left\{ \Big(a, (q_H)_{H\in\mathcal{H}}\Big) \in C^\infty({}^bS^*X) \times \prod_{H\in\mathcal{H}} \Psi^0(\Gamma_b(X)|_H \quad ; \quad \forall H \in \mathcal{H},\ a|_H = \sigma_b(q_H) \right\}. \quad (3.10)$$

The full symbol map:

$$\sigma_F : \Psi^0(\Gamma_b(X) \ni P \longmapsto \Big(\sigma_b(P), (P|_H)_{H\in\mathcal{H}}\Big) \in \mathcal{A}_{\mathcal{F}} \quad (3.11)$$

extends to the C^*-closures of the algebras and the assertion about the invertibility modulo compact operators amounts to the exactness of the sequence [12]:

$$0 \longrightarrow \mathcal{K}(X) \longrightarrow \overline{\Psi^0(\Gamma_b(X))} \xrightarrow{\sigma_F} \overline{\mathcal{A}_{\mathcal{F}}} \longrightarrow 0 \quad (3.12)$$

Then one set:

Definition 3.1 (Full Ellipticity) *An operator $D \in \Psi^0(\Gamma_b(X))$ is said to be fully elliptic if $\sigma_F(D)$ is invertible.*

We then recall the following result of Loya [14] (the statement also appears in [16]). Remember that b-Sobolev spaces $H_b^s(X)$ are defined using b-metrics and b-operators map continuously $H_b^m(X)$ to $H_b^{s-m}(X)$ for every s.

Theorem 3.2 ([14], Theorem 2.3) *An operator $D \in \Psi_b^0(X)$ is fully elliptic if and only if it is Fredholm on $H_b^s(X)$ for some s (and then for any s, with Fredholm index independent of s).*

For a given fully elliptic operator D, we denote by $\mathrm{Ind}_{\mathrm{Fred}}(D)$ its Fredholm index. We are going to express this number in terms of K-theory and clarify the relationship between the analytical index and full ellipticity on X using deformation groupoids. Let us star with the tangent groupoid

$$\Gamma_b(X)^{tan} := (G_c(\tilde{X}, (\rho_i))^{tan})|_{X\times[0,1]} = T_bX \bigsqcup \Gamma_b(X) \times (0,1] \rightrightarrows X \times [0,1]. \quad (3.13)$$

Now we introduce the two following saturated subspaces of $X \times [0,1]$:

$$X_{\mathcal{F}} := X \times [0,1] \setminus \partial X \times \{1\} \text{ and } X_\partial := X_{\mathcal{F}} \setminus \overset{\circ}{X} \times (0,1] = X \cup \partial X \times [0,1). \quad (3.14)$$

The **Fredholm b-groupoid** and the **noncommutative tangent space** of X are defined by

$$\Gamma_b(X)^{Fred} := \Gamma_b(X)^{tan}|_{X_{\mathcal{F}}} \text{ and } T_{nc}X := \Gamma_b(X)^{Fred}|_{X_\partial} \quad (3.15)$$

respectively. They are obviously KK-equivalent as one sees using the exact sequence:

$$0 \longrightarrow C^*(\overset{\circ}{X} \times \overset{\circ}{X} \times (0,1]) \longrightarrow C^*(\Gamma_b(X)^{\mathcal{F}}) \xrightarrow{r_F} C^*(T_{nc}X) \longrightarrow 0 \quad (3.16)$$

whose ideal is K-contractible. We then define the **Fredholm index morphism** by:

$$Ind_F^X = (r_1)_* \circ (r_F)_*^{-1} : K^0(T_{nc}X) \longrightarrow K^0(\overset{\circ}{X} \times \overset{\circ}{X}) \simeq \mathbb{Z}, \quad (3.17)$$

Following [7, Definition 10.4], we denote by $\mathrm{FE}(X)$ the group of order 0 fully elliptic operators modulo stable homotopy. Then the vocabulary above is justified by:

Proposition 3.3 *There exists a group isomorphism*

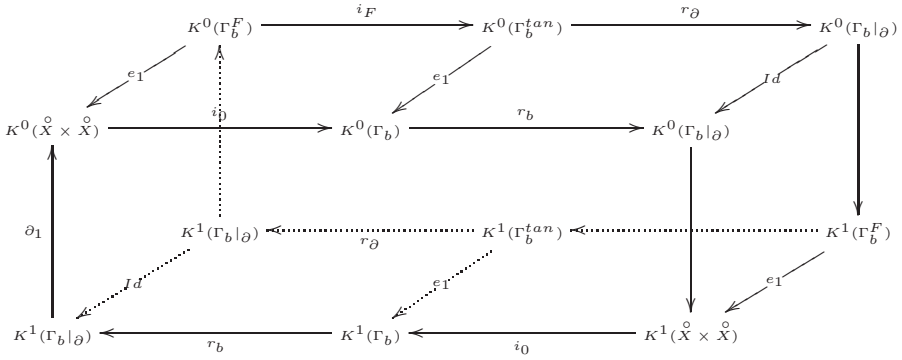
$$\sigma_{nc} : \mathrm{FE}(X) \longrightarrow K_0(C^*(\mathcal{T}_{nc}X)) \quad (3.18)$$

such that

$$r_0([\sigma_{nc}(D)]) = [\sigma_b(D)] \in K_0(C_0({}^bT^*X)) \quad \text{and} \quad Ind_F^X([\sigma_{nc}(D)]) = \mathrm{Ind}_{\mathrm{Fred}}(D), \quad (3.19)$$

where r_0 comes from the natural restriction map $C^*(\mathcal{T}_{nc}X) \to C_0({}^bT^*X)$.

This is proved by the method leading to [26, Theorem 4] and [7, Theorem 10.6] exactly in the same way. Also, this homotopy classification appears in [21], in which the K-homology of a suitable *dual manifold* is used instead of the K-theory of the noncommutative tangent space. Previous related results appeared in [12] for differential operators and using different algebras to classify their symbols.
The construction of the various index maps above is summarized into the commutative diagram:



(3.20)

3.3 Fredholm perturbation property

We are ready to define the Fredholm Perturbation Property [22] and its stably homotopic version.

Definition 3.4 *Let $D\in\Psi_b^m(X)$ be elliptic. We say that D satisfies:*

- *the Fredholm Perturbation Property ($\mathcal{FP}$) if there is $R\in\Psi_b^{-\infty}(X)$ such that $D+R$ is fully elliptic.*
- *the stably homotopic Fredholm Perturbation Property ($\mathcal{HFP}$) if there is a fully elliptic operator D' with $[\sigma_b(D')]=[\sigma_b(D)]\in K_0(C^*({}^bTX))$.*

We also say that X satisfies the *(stably homotopic) Fredholm Perturbation Property* if any elliptic b-operator on X satisfies ($(\mathcal{H})\mathcal{FP}$).

Property ($\mathcal{FP}$) is of course stronger than property ($\mathcal{HFP}$). In [22], Nistor characterized ($\mathcal{FP}$) in terms of the vanishing of an index in some particular cases. In [21], Nazaikinskii, Savin and Sternin characterized ($\mathcal{HFP}$) for arbitrary manifolds with corners using an index map associated with their dual manifold construction. We now rephrase the result of [21] in terms of deformation groupoids.

Theorem 3.5 *Let D be an elliptic b-pseudodifferential operator on a compact manifold with corners X. Then D satisfies ($\mathcal{HFP}$) if and only if $Ind_\partial([\sigma_b(D)])=0$.*
In particular if D satisfies ($\mathcal{FP}$) then its analytic indicial index vanishes.

Proof : Note that the Fredholm and the tangent groupoids are related by the exact sequence

$$0\longrightarrow C^*(\Gamma_b^{Fred}(X))\xrightarrow{i_F}C^*(\Gamma_b^{tan}(X))\xrightarrow{r_\partial}C^*(\Gamma_b(X)_{\partial X})\longrightarrow 0 \tag{3.21}$$

Then Proposition 3.3, together with this exact sequence and the commutative diagram:

$$\begin{array}{ccc} K_0(C^*(\Gamma_b^F)) & \xrightarrow[r_F]{\simeq} & K_0(C^*(T_{nc}X)) \\ \downarrow{\scriptstyle i_F} & & \downarrow{\scriptstyle r_0} \\ K_0(C^*(\Gamma_b^{tan})) & \xrightarrow[r_0]{\simeq} & K_0(C^*({}^bTX)) \end{array} \tag{3.22}$$

yields the result. □

Loosely speaking, this theorem tells us that the K-theory of $\Gamma_b(X)_{\partial X}$, or equivalently the one of $\Gamma_b(X)$ as we shall see later, is the receptacle for the obstruction to Fredholmness of elliptic symbols in the b-calculus. This is why we now focus on the understanding of these K-theory groups. If the result is well kwown in codimension less or equal to 1, the general case is far from understood. Meanwhile, we will also clarify the equivalent role played by $\Gamma_b(X)$ and $\Gamma_b(X)_{\partial X}$.

4 The conormal homology of a manifold with corners

In all this section, X is a manifold with embedded corners of codimension d, whose hyperfaces $H_1, \dots, \dots H_N$ are provided with defining functions $r_1, \dots, r_N$.

4.1 Definition of the homology

The one form $e_j = dr_j$ trivialises the conormal bundle of H_j for any $1 \leq j \leq N$. By convention, p-uples of integers $I = (i_1, \dots, i_p) \in \mathbb{N}^p$ are always labelled so that $1 \leq i_1 < \dots < i_p \leq N$. Let I be a p-uple, set

$$H_I = r_I^{-1}(\{0\}) = H_{i_1} \cap \dots \cap H_{i_p}. \tag{4.1}$$

and note $c(I)$ the set of open connected faces of codimension p included in H_I. Also, we denote by e_I the exterior product

$$e_I = e_{i_1}.e_{i_2}.\dots.e_{i_p}. \tag{4.2}$$

Let f be a face of codimension p and I the p-uple such that $f \in c(I)$. The conormal bundle $N(f)$ of f has a global basis given by the sections e_j, $j \in I$, and its orientations are identified with $\pm e_I$. For any integer $0 \leq p \leq d$, we denote by $C_p(X)$ the free $\mathbb{Z}$-module generated by

$$\{f \otimes \varepsilon \ ; \ f \in F_p, \ \varepsilon \text{ is an orientation of } N(f)\}. \tag{4.3}$$

Let $f \in F_p$, ϵ_f an orientation of $N(f)$ and $g \in F_{p-1}$ such that $f \subset \overline{g}$. The face f is characterized in $\overline{g}$ by the vanishing of a defining function $r_{i_{(g,f)}}$. Then the contraction $e_{i_{(g,f)}} \lrcorner \epsilon_f$ is an orientation of $N(g)$. Recall that the contraction $\lrcorner$ is defined by

$$e_i \lrcorner e_I = \begin{cases} 0 & \text{if } i \notin I \\ (-1)^{j-1} e_{I \setminus \{i\}} & \text{if } i \text{ is the } j^{\text{th}} \text{ coordinate of } I. \end{cases} \tag{4.4}$$

We then define $\delta_p : C_p(X) \to C_{p-1}(X)$ by

$$\delta_p(f \otimes \varepsilon_f) = \sum_{\substack{g \in F_{p-1}, \\ f \subset \overline{g}}} g \otimes e_{i_{(g,f)}} \lrcorner \varepsilon_f. \tag{4.5}$$

It is not hard to check directly that $(C_*(X), \delta_*)$ is a differential complex. Actually, δ_* is the component of degree -1 of another natural differential map $\delta^{\text{pcn}} = \sum_{k \geq 0} \delta^{2k+1}$, which eventually produces a quasi-isomorphic differential complex. Details are provided in the last section.

We define the *conormal homology* of X as the homology of $(C_*(X), \delta_*)$, and we note

$$H_p^{\text{cn}}(X) := H_p(C_*(X), \delta_*). \tag{4.6}$$

Also, the graduation of the conormal homology into even and odd degree, called here *periodic conormal homology*, will be used and we note

$$H_0^{\text{pcn}}(X) = \oplus_{p \geq 0} H_{2p}^{\text{cn}}(X) \text{ and } H_1^{\text{pcn}}(X) = \oplus_{p \geq 0} H_{2p+1}^{\text{cn}}(X). \tag{4.7}$$

4.2 Examples

The determination of the groups $H_*^{\text{cn}}(X)$ is completely elementary in all concrete cases. In the following examples, it is understood that faces f arise with the orientation given by e_I if $f \in c(I)$.

Example 4.1

- *Assume that X has no boundary. Then $H_0^{\text{pcn}}(X) = H_0^{\text{cn}}(X) \simeq \mathbb{Z}$, $H_1^{\text{pcn}}(X) = 0$.*
- *Assume that X has a boundary with n connected components. Then $H_0^{\text{pcn}}(X) = 0$ and $H_1^{\text{pcn}}(X) = H_1^{\text{cn}}(X) \simeq \mathbb{Z}^{n-1}$. More precisely, if we set $F_1 = \{f_1, ..., f_n\}$ then $\{f_1 - f_2, f_2 - f_3, ..., f_{n-1} - f_n\}$ provides a basis of $\ker \delta_1$.*
- *Assume that X has codimension 2 and that ∂X is connected. Then $H_0^{\text{pcn}}(X) = H_2^{\text{cn}}(X) = \ker \delta_2 \simeq \mathbb{Z}^k$, where all nonnegative integers k can arise. For instance, consider the unit closed ball B in $\mathbb{R}^3$, cut $k+1$ small disjoint disks out of its boundary and glue two copies of such spaces along the pairs of cut out disks. We get a space X satisfying the statement: the boundaries $s_0, \dots, s_k$ of the original disks provide a basis of F_2 and the family $s_0 - s_j$, $1 \leq j \leq k$ a basis of $\ker \delta_2$. Finally, $([0, +\infty))^2$ provides an example with $k = 0$.*

- *Consider the cube* $X = [0,1]^3$.
  1. *We have* $H_0^{\rm pcn}(X) = 0$ *and* $H_1^{\rm pcn}(X) = H_3^{\rm cn}(X) \simeq \mathbb{Z}$.
  2. *Remove a small open cube into the interior of* X *and call the new space* Y. *Then*
  $$H_0^{\rm pcn}(Y) = 0 \text{ and } H_1^{\rm pcn}(Y) = H_3^{\rm cn}(Y) \oplus H_1^{\rm cn}(Y) \simeq \mathbb{Z}^2 \oplus \mathbb{Z}.$$
  3. *Remove a small open ball into the interior of* X *and call the new space* Z. *Then*
  $$H_0^{\rm pcn}(Z) = 0 \text{ and } H_1^{\rm pcn}(Y) = H_3^{\rm cn}(Y) \oplus H_1^{\rm cn}(Y) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

4.3 Long exact sequence in conormal homology

We define a filtration of X by open submanifolds with corners by setting:

$$X_m = \bigcup_{f \in F_k,\ k \leq m} f, \qquad 0 \leq m \leq d. \tag{4.8}$$

This leads to differential complexes $(C_*(X_m), \delta)$ for $0 \leq m \leq d$. We can also filtrate the differential complex $(C_*(X), \delta)$ by the codimension of faces:

$$F_m(C_*(X)) = \bigoplus_{k=0}^{m} C_k(X). \tag{4.9}$$

There is an obvious identification $C_*(X_m) \simeq F_m(C_*(X))$ and we thus consider $(C_*(X_m), \delta)$ as a subcomplex of $(C_*(X), \delta)$, with quotient complex denoted by $(C_*(X, X_m), \delta)$. The quotient module is also naturally embedded in $C_*(X)$:

$$C_*(X, X_m) = C_*(X)/C_*(X_m) \simeq \bigoplus_{k=m+1}^{d} C_k(X) \subset C_*(X). \tag{4.10}$$

The embedding, denoted by ρ, is a section of the quotient map. The short exact sequence:

$$0 \longrightarrow C_*(X_m) \longrightarrow C_*(X) \longrightarrow C_*(X, X_m) \longrightarrow 0 \tag{4.11}$$

induces a long exact sequence in conormal homology:

$$\cdots \xrightarrow{\partial_{p+1}} H_p^{\rm cn}(X_m) \longrightarrow H_p^{\rm cn}(X) \longrightarrow H_p^{\rm cn}(X, X_m) \xrightarrow{\partial_p} H_{p-1}^{\rm cn}(X_m) \longrightarrow \cdots \tag{4.12}$$

and we need to precise the connecting homomorphism.

Proposition 4.2 *Let* $[c] \in H_p^{\rm cn}(X, X_m)$. *Then*

$$\partial_p [c] = [\delta(\rho(c))]. \tag{4.13}$$

Proof : Since c is by assumption a cycle in $(C_*(X, X_m), \delta)$, the chain $\rho(c)$ has a boundary made of faces contained in X_m. The result follows. □

Remarks 4.3
• *We can replace* X *by* X_l *and quotient the exact sequence* (4.11) *by* $C_*(X_q)$ *for any integers* l, m, q *such that* $0 \leq q \leq m \leq l \leq d$. *This leads to long exact sequences:*

$$\cdots \xrightarrow{\partial} H_p^{\rm cn}(X_m, X_q) \longrightarrow H_p^{\rm cn}(X_l, X_q) \longrightarrow H_p^{\rm pcn}(X_l, X_m) \xrightarrow{\partial} H_{p-1}^{\rm pcn}(X_m, X_q) \longrightarrow \cdots \tag{4.14}$$

whose connecting homomorphisms are again given by the formula of Proposition 4.2.
• *If we split the conormal homology into even and odd periodic groups, then the long exact sequence* (4.12) *becomes a six term exact sequence:*

$$\begin{array}{ccccc} H_0^{\rm pcn}(X_m) & \longrightarrow & H_0^{\rm pcn}(X) & \longrightarrow & H_0^{\rm pcn}(X, X_m) \\ \uparrow{\scriptstyle \partial^1} & & & & \downarrow{\scriptstyle \partial^0} \\ H_1^{\rm pcn}(X, X_m) & \longleftarrow & H_1^{\rm pcn}(X) & \longleftarrow & H_1^{\rm pcn}(X_m) \end{array} \tag{4.15}$$

where ∂^0, ∂^1 are given by the direct sum in even/odd degrees of the maps ∂_* of Proposition 4.2.
• *We can replace X_m in the exact sequence (4.11) by an open saturated submanifold $U \subset X_m$, that is, an open subset of X consisting of an union of faces. This gives in the same way a subcomplex $(C_*(U), \delta)$ of $(C_*(X), \delta)$ and a section $\rho : C_*(X, U) \to C_*(X)$ allowing to state Proposition 4.2 verbatim. More generally, if U is any open submanifold of X and $\widetilde{U}$ denotes the smallest open saturated submanifold containing U, then any face f of U is contained in a unique face $\widetilde{f}$ of X and an oriention of $N(f)$ determines an orientation of $N(\widetilde{f})$. This gives rise to a quasi-isomorphism $C_*(U) \to C_*(\widetilde{U})$.*

Finally, assume that $d \geq 1$. Since X is connected, the map $\delta_1 : C_1(X) \to C_0(X)$ is surjective, which implies by Proposition 4.2 the surjectivity of the connecting homomorphism $\partial^1 : H_1^{\mathrm{pcn}}(X, X_0) \to H_0^{\mathrm{pcn}}(X_0)$. This fact and $H_1^{\mathrm{pcn}}(X_0) = 0$ gives, using (4.15), the useful corollary:

Corollary 4.4 *For any connected manifold with corners X of codimension $d \geq 1$ the canonical morphism $H_0^{\mathrm{pcn}}(X) \to H_0^{\mathrm{pcn}}(X, X_0)$ is an isomorphism.*

4.4 Torsion free in low codimensions

Here we will show that up to codimension 2 the conormal homology groups (and later on the K-theory groups) are free abelian groups.

Lemma 4.5 *Let X be of arbitrary codimension and assume that ∂X has l connected components. Then $H_1^{\mathrm{cn}}(X) \simeq \mathbb{Z}^{l-1}$.*

Proof : For any face f, denote by $cc(f)$ the connected component of ∂X containing f. It is obvious that $\ker \delta_1$ is generated by the differences $f - g$ where f, g run through F_1. Let $f, g \in F_1$ such that $cc(f) = cc(g)$. Then there exist $f_0, \ldots, f_l \in F_1$ such that $f = f_0$, $g = f_l$ and $\overline{f_i} \cap \overline{f_{i+1}} \neq \emptyset$ for any i. Therefore for any i, there exists $f_{i,i+1} \in F_2$ such that $\delta_2(f_{i,i+1}) = f_i - f_{i+1}$, hence $f - g = \delta_2(\sum f_{i,i+1})$ is a boundary in conormal homology.

Now assume that $cc(f) \neq cc(g)$. By the previous discussion, we also have $[f - g] = [f' - g'] \in H_1^{\mathrm{cn}}(X)$ for any $f', g' \in F_1$ such that $f' \subset cc(f)$ and $g' \subset cc(g)$. Therefore, pick up one hyperface in each connected component of ∂X, call them $f_1, \ldots, f_l$, and set $\alpha_i = [f_1 - f_i] \in H_1^{\mathrm{cn}}(X)$ for $i \in \{2, \ldots, l\}$. It is obvious that $(\alpha_i)_{2 \leq i \leq l}$ generates $H_1^{\mathrm{cn}}(X)$. So, consider integers $x_2, \ldots, x_l$ such that

$$\sum_{i=2}^{l} x_i \alpha_i = 0.$$

In other words, there exists $x \in C_2(X)$ such that

$$(\sum_{i=2}^{l} x_i) f_1 - \sum_{i=2}^{l} x_i f_i = \delta_2(x). \tag{4.16}$$

For any $p \geq 1$ and $2 \leq j \leq l$ denote by $\pi_j : C_p(X) \to C_p(X)$ the map defined by $\pi_j(h) = h$ if $h \subset cc(f_j)$ and $\pi_j(h) = 0$ otherwise. All π_is commute with δ_*, hence (4.16) gives

$$\forall 2 \leq j \leq l, \quad x_j f_j = \delta_2(\pi_j(x)).$$

Since $\delta_1(f_j) = \overset{\circ}{X} \neq 0$, we conclude $x_j = 0$ for all j. □

Theorem 4.6 *Let assume that X is connected and has codimension $d \leq 2$. Then $H_*^{\mathrm{pcn}}(X)$ is a free abelian group.*

Proof : This is essentially a compilation of previous examples and computations. The first two cases in Example 4.2 give the result for $d = 0$ and $d = 1$. If X is of codimension 2, then the third case in Example 4.2 says that $H_0^{\mathrm{pcn}}(X)$ is free. In codimension 2 again, we have $H_1^{\mathrm{pcn}}(X) = H_1^{\mathrm{cn}}(X)$, hence we are done by Lemma 4.5. □

Remark 4.7 *If $\mathrm{codim}(X) = 3$, then $H_1^{\mathrm{pcn}}(X) = H_1^{\mathrm{cn}}(X) \oplus H_3^{\mathrm{cn}}(X)$. Since $H_3^{\mathrm{cn}}(X) = \ker \delta_3$, Lemma 4.5 also gives that $H_1^{\mathrm{pcn}}(X)$ is free. The combinatorics needed to prove that $H_2^{\mathrm{cn}}(X)$ - and therefore $H_0^{\mathrm{pcn}}(X)$- is free are much more involved. The torsion of conormal homology for manifolds of arbitrary codimension will be studied somewhere else.*

4.5 Künneth Formula for Conormal homology

Taking advantage of the previous paragraph, we consider a product $X = X_1 \times X_2$ of two manifolds with corners, one of them being of codimension ≤ 2. It is understood that the defining functions used for X are obtained by pulling back the ones used for X_1 and X_2. The tensor product $(\widehat{C}_*, \widehat{\delta})$ of the conormal complexes of X_1 and X_2 is given by

$$\widehat{C}_p = \bigoplus_{s+t=p} C_s(X_1) \otimes C_t(X_2) \quad \text{and} \quad \widehat{\delta}(x \otimes y) = \delta(x) \otimes y + (-1)^t x \otimes \delta(y) \tag{4.17}$$

where $x \in C_t(X_1)$ in the second formula. We have an isomorphism of differential complexes:

$$(\widehat{C}_*, \widehat{\delta}) \simeq (C_*(X), \delta). \tag{4.18}$$

It is given by the map

$$\Psi_p : \widehat{C}_p = \bigoplus_{s+t=p} C_s(X_1) \otimes C_t(X_2) \longrightarrow C_p(X) \tag{4.19}$$

defined by:

$$(f \otimes \epsilon_f) \otimes (g \otimes \epsilon_g) \longmapsto (f \times g) \otimes \epsilon_f \cdot \epsilon_g, \tag{4.20}$$

where we did not distinguish differential forms on X_j and their pull-back to X via the canonical projections and $\cdot$ denotes again the exterior product. Since $H_*^{\mathrm{cn}}(X_j)$ is torsion free for $j = 1$ or 2 by assumption, we get by Künneth Theorem:

$$H_p(\widehat{C}_*, \hat{\delta}) = \bigoplus_{s+t=p} H_s^{\mathrm{cn}}(X_1) \otimes H_t^{\mathrm{cn}}(X_2). \tag{4.21}$$

Therefore:

Proposition 4.8 (Künneth Formula) *Assume that $X = X_1 \times X_2$ with one factor at least of codimension ≤ 2. Then we have:*

$$H_0^{\mathrm{pcn}}(X) \simeq H_0^{\mathrm{pcn}}(X_1) \otimes H_0^{\mathrm{pcn}}(X_2) \oplus H_1^{\mathrm{pcn}}(X_1) \otimes H_1^{\mathrm{pcn}}(X_2), \tag{4.22}$$

$$H_1^{\mathrm{pcn}}(X) \simeq H_0^{\mathrm{pcn}}(X_1) \otimes H_1^{\mathrm{pcn}}(X_2) \oplus H_1^{\mathrm{pcn}}(X_1) \otimes H_0^{\mathrm{pcn}}(X_2) \tag{4.23}$$

The following straightforward corollary will be useful later on:

Corollary 4.9 *If $X = \Pi_i X_i$ is a finite product of manifold with corners X_i with $codim(X_i) \leq 2$, then the groups $H_*^{\mathrm{pcn}}(X)$ are torsion free.*

The exact same arguments as above work to show that the Kunneth formula holds in full generality for conormal homology with rational coefficients, *i.e.* for $H_*^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. We state the proposition as we will use it later:

Proposition 4.10 (Künneth Formula with rational coefficients) *For $X = X_1 \times X_2$ we have:*

$$H_0^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq (H_0^{\mathrm{pcn}}(X_1) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes (H_0^{\mathrm{pcn}}(X_2) \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (H_1^{\mathrm{pcn}}(X_1) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes (H_1^{\mathrm{pcn}}(X_2) \otimes_{\mathbb{Z}} \mathbb{Q}), \tag{4.24}$$

$$H_1^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq (H_0^{\mathrm{pcn}}(X_1) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes (H_1^{\mathrm{pcn}}(X_2) \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (H_1^{\mathrm{pcn}}(X_1) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes (H_0^{\mathrm{pcn}}(X_2) \otimes_{\mathbb{Z}} \mathbb{Q}) \tag{4.25}$$

5 The computation of $K_*(\mathcal{K}_b(X))$

We keep all the notations and conventions of Section 4. In particular, the defining functions induce a trivialisation of the conormal bundle of any face f:

$$N(f) \simeq f \times E_f, \tag{5.1}$$

in which the p-dimensional real vector space E_f inherits a basis $b_f = (e_i)_{i \in I}$, where I is characterized by $f \in c(I)$. These data induce an isomorphism

$$\Gamma_b(X)|_f \simeq C^*(\mathcal{C}(f) \times E_f) \tag{5.2}$$

where $\mathcal{C}(f)$ denotes the pair groupoid over f, as well as a linear isomorphism $\varphi_f : \mathbb{R}^p \to E_f$.

Also, the filtration (4.8) gives rise to the following filtration of the C^*-algebra $\mathcal{K}_b(X) = C^*(\Gamma_b(X))$ by ideals:

$$\mathcal{K}(L^2(\overset{\circ}{X})) = A_0 \subset A_1 \subset \ldots \subset A_d = A = \mathcal{K}_b(X), \tag{5.3}$$

with $A_m = C^*(\Gamma(X)|_{X_m})$ for any $0 \leq m \leq d$. The isomorphisms (5.2) induce

$$A_m/A_{m-1} \simeq C^*(\Gamma|_{X_m \setminus X_{m-1}}) \simeq \bigoplus_{f \in F_m} C^*(\mathcal{C}(f) \times E_f). \tag{5.4}$$

5.1 The first differential of the spectral sequence for $K_*(A)$

The K-theory spectral sequence $(E^r_{*,*}, d^r_{*,*})_{r \geq 1}$ associated with (5.3) [27, 11] converges to:

$$E^\infty_{p,q} = K_{p+q}(A_p)/K_{p+q}(A_{p-1}). \tag{5.5}$$

Here we have set $K_n(A) = K_0(A \otimes C_0(\mathbb{R}^n))$ for any C^*-algebra A. By construction, all the terms $E^r_{p,2q+1}$ vanish and by Bott periodicity, $E^r_{p,2q} \simeq E^r_{p,0}$. Also, all the differentials $d^{2r}_{p,q}$ vanish. By definition:

$$d^1_{p,q} : E^1_{p,q} = K_{p+q}(A_p/A_{p-1}) \longrightarrow E^1_{p-1,q} = K_{p+q-1}(A_{p-1}/A_{p-2}) \tag{5.6}$$

is the connecting homomorphism of the short exact sequence:

$$0 \longrightarrow A_{p-1}/A_{p-2} \longrightarrow A_p/A_{p-2} \longrightarrow A_p/A_{p-1} \longrightarrow 0. \tag{5.7}$$

By (5.4), we get isomorphisms:

$$E^1_{p,q} \simeq \bigoplus_{f \in F_p} K_{p+q}(C^*(\mathcal{C}(f) \times E_f)). \tag{5.8}$$

Since the real vector space E_f has dimension p, the groups $E^1_{p,q}$ vanish for odd q and for even q, we have after applying Bott periodicity, $E^1_{p,q} \simeq \mathbb{Z}^{\#F_p}$.

Melrose and Nistor [15, Theorem 9] already achieved the computation of $d^1_{*,*}$. In order to relate the terms $E^2_{*,*}$ with the elementary defined conormal homology, we reproduce their computation in a slightly different way. Our approach is based on the next two lemmas.

Lemma 5.1 *Let $\mathbb{R}_+ \rtimes \mathbb{R}$ be the groupoid of the action of $\mathbb{R}$ onto $\mathbb{R}_+$ given by*

$$t.\lambda = te^\lambda, \qquad t \in \mathbb{R}_+,\ \lambda \in \mathbb{R}. \tag{5.9}$$

The element $\alpha \in KK_1(C^*(\mathbb{R}), C^*(\mathbb{R}_+^*))$ associated with the exact sequence

$$0 \longrightarrow C^*(\mathcal{C}(\mathbb{R}_+^*)) \longrightarrow C^*(\mathbb{R}_+ \rtimes \mathbb{R}) \longrightarrow C^*(\mathbb{R}) \longrightarrow 0 \tag{5.10}$$

is a KK-equivalence.

Proof : By the Thom-Connes isomorphism, the C^*-algebras $C^*(\mathbb{R}_+ \rtimes \mathbb{R})$ and $C^*(\mathbb{R}_+ \times \mathbb{R})$ are KK-equivalent. The latter being K-contractible, the result follows. □

Lemma 5.2 *Let $\mathbb{R}_+ \rtimes_i \mathbb{R}^p$ be the groupoid given by the action of the i^{th} coordinate of $\mathbb{R}^p$ on $\mathbb{R}_+$ by (5.9). Let $\alpha_{i,p} \in KK_1(C^*(\mathbb{R}^p), C^*(\mathbb{R}^{p-1}))$ be the KK-element induced by the exact sequence*

$$0 \longrightarrow C^*(\mathcal{C}(\mathbb{R}_+^*) \times \mathbb{R}^{p-1}) \longrightarrow C^*(\mathbb{R}_+ \rtimes_i \mathbb{R}^p) \longrightarrow C^*(\mathbb{R}^p) \longrightarrow 0. \tag{5.11}$$

Then for all $1 \leq i \leq p$ we have

$$\alpha_{i,p} = (-1)^{i-1}\alpha_{1,p} \text{ and } \alpha_{1,p} = \sigma_{C^*(\mathbb{R}^{p-1})}(\alpha), \tag{5.12}$$

where $\sigma_D : K_*(A, B) \to K_*(A \otimes D, B \otimes D)$ denotes the Kasparov suspension map.

Proof : Let τ be a permutation of $\{1,2,\dots,p\}$ and $i \in \{1,\dots,p\}$. We denote in the same way the corresponding automorphisms of $\mathbb{R}^p$ and $C^*(\mathbb{R}^p)$. We have a groupoid isomorphism

$$\widetilde{\tau} : \mathbb{R}_+ \rtimes_i \mathbb{R}^p \xrightarrow{\simeq} \mathbb{R}_+ \rtimes_{\tau(i)} \mathbb{R}^p$$

and if we denote by τ_i the automorphism of $\mathbb{R}^{p-1}$ obtained by removing the i^{th} factor in the domain of τ and the $\tau(i)^{\text{th}}$ factor in the range of τ, we get a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(\mathcal{C}(\mathbb{R}_+^*) \times \mathbb{R}^{p-1})) & \longrightarrow & C^*(\mathbb{R}_+ \rtimes_i \mathbb{R}^p) & \longrightarrow & C^*(\mathbb{R}^p) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau_i} & & \downarrow{\scriptstyle \widetilde{\tau}} & & \downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & C^*(\mathcal{C}(\mathbb{R}_+^*) \times \mathbb{R}^{p-1})) & \longrightarrow & C^*(\mathbb{R}_+ \rtimes_{\tau(i)} \mathbb{R}^p) & \longrightarrow & C^*(\mathbb{R}^p) & \longrightarrow & 0 \end{array} \tag{5.13}$$

It follows that

$$\alpha_{\tau(i),p} = [\tau^{-1}] \otimes \alpha_{i,p} \otimes [\tau_i] \in KK_1(C^*(\mathbb{R}^p), \mathcal{K} \otimes C^*(\mathbb{R}^{p-1})). \tag{5.14}$$

Taking $\tau = (1,i)$, we get $\tau = \tau^{-1}$ and $\tau_i = \mathrm{id}$, so that $\alpha_{i,p} = [\tau] \otimes \alpha_{1,p}$. Moreover, observe that for any j,

$$[(j-1,j)] = 1_{j-2} \otimes [f] \otimes 1_{p-j} \in \mathbb{K}(C^*(\mathbb{R}^p), C^*(\mathbb{R}^p)) \tag{5.15}$$

where $[f] = -1 \in KK(C^*(\mathbb{R}^2), C^*(\mathbb{R}^2))$ is the class of the flip automorphism and we have used the identification

$$C^*(\mathbb{R}^p) = C^*(\mathbb{R}^{j-2}) \otimes C^*(\mathbb{R}^2) \otimes C^*(\mathbb{R}^{p-j}).$$

Using

$$(1,i) = (1,2).(2,3)\dots(i-1,i)$$

now gives $[\tau] = (-1)^{i-1}$. Factorizing $C^*(\mathbb{R}^{p-1})$ on the right in the sequence (5.11) for $i = 1$ gives the assertion $\alpha_{1,p} = \sigma_{C^*(\mathbb{R}^{p-1})}(\alpha)$. □

Using the canonical isomorphism $KK_1(C^*(\mathbb{R}), C^*(\mathbb{R}_+^*)) \simeq KK_1(C_0(\mathbb{R}), \mathbb{C})$, we can define a generator β of $K_1(C_0(\mathbb{R}))$ by

$$\beta \otimes \alpha = +1. \tag{5.16}$$

For any $f \in F_p$ we then obtain a generator β_f of $K_p(C_0(E_f))$ by

$$\beta_f = (\varphi_f)_*(\beta^p) \in K_p(C_0(E_f)) \tag{5.17}$$

where β^p is the external product:

$$\beta^p = \beta \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \beta \in K_p(C_0(\mathbb{R}^p)). \tag{5.18}$$

Picking up rank one projectors p_f in $C^*(\mathcal{C}(f))$, we get a basis of the free $\mathbb{Z}$-module $E^1_{p,0}$:

$$(p_f \otimes \beta_f)_{f \in F_p}. \tag{5.19}$$

Bases of $E^1_{p,q}$ for all even q are deduced from the previous one by applying Bott periodicity.

Now consider faces $f \in F_p$ and $g \in F_{p-1}$ such that $f \subset \partial\overline{g}$. The p and $p-1$ uples I, J such that $f \in c(I)$ and $g \in c(J)$ differ by exactly one index, say the j^{th}, and we define

$$\sigma(f,g) = (-1)^{j-1}. \tag{5.20}$$

Introduce the exact sequence

$$0 \longrightarrow C^*(\mathcal{C}(f \times \mathbb{R}_+^*) \times E_g) \longrightarrow C^*(\mathcal{C}(f) \times (\mathbb{R}_+ \rtimes_j E_f)) \longrightarrow C^*(\mathcal{C}(f) \times E_f) \longrightarrow 0, \tag{5.21}$$

where $\mathbb{R}_+ \rtimes_j E_f$ denotes the transformation groupoid where the j^{th} coordinate (only) of E_f acts on $\mathbb{R}_+$ by (5.9) again. We note

$$\partial_{f,g} : K_p(C^*(\mathcal{C}(f) \times E_f)) \longrightarrow K_{p-1}(C^*(\mathcal{C}(g) \times E_g))$$

the connecting homomorphism associated with (5.21), followed by the unique KK-equivalence

$$C^*(\mathcal{C}(f \times \mathbb{R}_+^*)) \longrightarrow C^*(\mathcal{C}(g)) \tag{5.22}$$

provided by any tubular neighborhood of f into g.

Proposition 5.3 *With the notation above, we get*

$$\partial_{f,g}(p_f \otimes \beta_f) = \sigma(f,g).p_g \otimes \beta_g. \tag{5.23}$$

Proof : Identify $E_f \simeq \mathbb{R}^p$ and $E_g \simeq \mathbb{R}^{p-1}$ using b_f, b_g and apply Lemmas 5.2 and 5.1. □

We can now achieve the determination of $d^1_{*,*}$.

Theorem 5.4 *We have*

$$\forall f \in F_p, \qquad d^1_{p,0}(p_f \otimes \beta_f) = \sum_{\substack{g \in F_{p-1} \\ f \subset \partial \overline{g}}} \sigma(f,g) p_g \otimes \beta_g. \tag{5.24}$$

Proof : For $p = 0$, we have $F_{p-1} = \emptyset$ and $d^1_{p,0} = 0$, the result follows. For $p \geq 1$, we recall that

$$d^1_{p,0} : \oplus_{f \in F_p} K_p(C^*(\mathcal{C}(f) \times E_f)) \longrightarrow \oplus_{g \in F_{p-1}} K_{p-1}(C^*(\mathcal{C}(g) \times E_g)). \tag{5.25}$$

is the connecting homomorphism in K-theory of the exact sequence (5.7). We obviously have

$$d^1_{p,0}(p_f \otimes \beta_f) = \sum_{g \in F_{p-1}} \partial_g(p_f \otimes \beta_f) \tag{5.26}$$

where ∂_g is the connecting homomorphism in K-theory of the exact sequence

$$0 \longrightarrow C^*(\Gamma|_g) \longrightarrow C^*(\Gamma|_{g \cup f}) \longrightarrow C^*(\Gamma|_f) \longrightarrow 0. \tag{5.27}$$

If $f \not\subset \partial \overline{g}$ then the sequence splits and $\partial_g(p_f \otimes \beta_f) = 0$. Let $g \in F_{p-1}$ be such that $f \subset \partial \overline{g}$. Let $\mathcal{U}$ be an open neighborhood of f in X such that there exists a diffeomorphism

$$\mathcal{U}_g := \mathcal{U} \cap g \longrightarrow f \times (0, +\infty), \qquad x \longmapsto (\phi(x), r_{i_g}(x)), \tag{5.28}$$

where r_{i_g} is the defining function of f in $\overline{g}$. This yields a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(\Gamma|_{\mathcal{U}_g}) & \longrightarrow & C^*(\Gamma|_{\mathcal{U}_g \cup f}) & \longrightarrow & C^*(\Gamma|_f) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota} \hookrightarrow & & \downarrow \hookrightarrow & & \downarrow = & & \\ 0 & \longrightarrow & C^*(\Gamma|_g) & \longrightarrow & C^*(\Gamma|_{g \cup f}) & \longrightarrow & C^*(\Gamma|_f) & \longrightarrow & 0 \end{array} \tag{5.29}$$

whose upper sequence coincides with (5.21) using (5.28). This implies

$$\partial_g = \partial_{f,g}. \tag{5.30}$$

The result follows by Proposition 5.3. □

The map $d^1_{p,q}$, q even, is deduced from $d^1_{p,0}$ by Bott periodicity. We are ready to relate the E^2 terms with conormal homology.

Corollary 5.5 *For every $p \in \{1, ..., d\}$ there are isomorphisms*

$$\phi^i_{p,1} : H^{\mathrm{pcn}}_{\mathrm{i}}(X_p, X_{p-1}) \longrightarrow K_i(A_p/A_{p-1}), \qquad \{0,1\} \ni i \equiv p \mod 2, \tag{5.31}$$

such that the following diagram commute

$$\begin{array}{ccc} H^{\mathrm{pcn}}_{\mathrm{i}}(X_p, X_{p-1}) & \xrightarrow{\phi^i_{p,1}} & K_p(A_p/A_{p-1})) \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle d^1_{p,0}} \\ H^{\mathrm{pcn}}_{1-\mathrm{i}}(X_{p-1}, X_{p-2}) & \xrightarrow{\phi^{1-i}_{p-1,1}} & K_{p-1}(A_{p-1}/A_{p-2})) \end{array} \tag{5.32}$$

where ∂ stands for the connecting morphism in conormal homology.

Proof : If $i \equiv p \mod 2$ then $H^{\mathrm{pcn}}_{\mathrm{i}}(X_p, X_{p-1}) = C_p(X)$ and $\partial = \delta_p : C_p(X) \longrightarrow C_{p-1}(X)$. We define (5.31) by $\phi^i_{p,1}(f \otimes \epsilon_f) = p_f \otimes \beta_f$ and Theorem 5.4 gives the commutativity of (5.32). □

In other words, the map $f \otimes \epsilon_f \mapsto p_f \otimes \beta_f$ induces a isomorphism

$$H^{\mathrm{cn}}_p(X) \simeq E^2_{p,0}. \tag{5.33}$$

It would be very interesting to compute the higher differentials $d^{2r+1}_{p,0}$.

5.2 The final computation for $K_*(\mathcal{K}_b(X))$ in terms of conormal homology

Before getting to the explicit computations and to the analytic corollaries in term of these, let us give a simple but interesting result. It is about the full understanding of the six term exact sequence in K-theory of the fundamental sequence

$$0 \longrightarrow \mathcal{K}(X) \xrightarrow{i_0} \mathcal{K}_b(X) \xrightarrow{r} \mathcal{K}_b(\partial X) \longrightarrow 0. \tag{5.34}$$

Proposition 5.6 *For a connected manifold with corners X of codimension greater or equal to one the induced morphism by r in K_0, $r : K_0(\mathcal{K}_b(X)) \to K_0(\mathcal{K}_b(\partial X))$, is an isomorphism. Equivalently*

1. *The morphism $i_F : K_0(\mathcal{K}) \cong \mathbb{Z} \to K_0(\mathcal{K}_b(X))$ is the zero morphism.*
2. *The connecting morphism $K_1(\mathcal{K}_b(\partial X)) \to K_0(\mathcal{K}) \cong \mathbb{Z}$ is surjective.*

Proof : Let X be a connected manifold with corners of codimension d. With the notations of the last section, the sequence 5.34 correspond to the canonical sequence

$$0 \longrightarrow A_0 \longrightarrow A_d \longrightarrow A_d/A_0 \longrightarrow 0.$$

We will prove that the connecting morphism $K_1(A_d/A_0) \to K_0(A_0) \cong \mathbb{Z}$ is surjective. The proof will proceed by induction, the case $d = 1$ immediately satisfies this property. So let us assume that the connecting morphism $K_1(A_{d-1}/A_0) \to K_0(A_0)$ associated to the short exact sequence

$$0 \longrightarrow A_0 \longrightarrow A_{d-1} \longrightarrow A_{d-1}/A_0 \longrightarrow 0.$$

is surjective. Consider now the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & A_d/A_{d-1} & \longrightarrow & A_d/A_{d-1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A_0 & \longrightarrow & A_d & \longrightarrow & A_d/A_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A_0 & \longrightarrow & A_{d-1} & \longrightarrow & A_{d-1}/A_0 & \longrightarrow & 0.
\end{array} \tag{5.35}$$

By applying the six-term short eaxt sequence in K-theory to it we obtain that the following diagram, where ∂_d and ∂_{d-1} are the connecting morphisms associated to the middle and to the bottom rows respectively,

$$\begin{array}{ccc}
K_(A_d/A_0) & & \\
\uparrow & \searrow^{\partial_d} & \\
K_1(A_{d-1}/A_0) & \xrightarrow[\partial_{d-1}]{} & K_0(A_0)
\end{array}$$

is commutative. Hence, by inductive hypothesis, we obtain that ∂_d is surjective. □

Remark 5.7 *Roughly speaking, the previous proposition tells us that the analytical index of a fully elliptic element carries no information about its Fredholm index, this information being completely contained in some element of $K_1(\mathcal{K}_b(\partial X))$.*

We have next our main K-theoretical computation:

Theorem 5.8 *Let X be a finite product of manifolds with corners of codimension less or equal to three. There are natural isomorphisms*

$$H_0^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow[\cong]{\phi_X} K_0(\mathcal{K}_b(X))) \otimes_{\mathbb{Z}} \mathbb{Q} \quad \textit{and} \qquad H_1^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow[\cong]{\phi_X} K_1(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{5.36}$$

In the case X contains a factor of codimension at most two or X is of codimension three the result holds even without tensoring by $\mathbb{Q}$.

Proof :

1A. $Codim(X) = 0$**:** The only face of codimension 0 is $\overset{\circ}{X}$ (we are always assuming X to be connected). The isomorphism

$$H_0^{\mathrm{cn}}(X_0) \xrightarrow{\phi_0} K_0(A_0)$$

is simply given by sending $\overset{\circ}{X}$ to the rank one projector $p_{\overset{\circ}{X}}$ chosen in section 5.1.

1B. $Codim(X) = 1$**:** Consider the canonical short exact sequence

$$0 \longrightarrow A_0 \longrightarrow A_1 \longrightarrow A_1/A_0 \longrightarrow 0$$

That gives, since $d^1_{1,0}$ is surjective, the following exact sequence in K-theory

$$0 \longrightarrow K_1(A_1) \longrightarrow K_1(A_1/A_0) \xrightarrow{d^1_{1,0}} K_0(A_0) \longrightarrow 0$$

from which $K_1(A_1) \cong \ker d^1_{1,0}$ and $K_0(A_1) = 0$ (since $K_0(A_1/A_0) = 0$ by a direct computation for K-theory or for conormal homology). By theorem 5.4, corollary 5.5, we have the following commutative diagram

$$\begin{array}{ccc} K_1(A_1/A_0) & \xrightarrow{d^1_{1,0}} & K_0(A_0) \\ {\scriptstyle \phi_{1,0}}\uparrow{\scriptstyle \cong} & & {\scriptstyle \phi_0}\uparrow{\scriptstyle \cong} \\ H_1^{\mathrm{pcn}}(X_1 \setminus X_0) & \xrightarrow[\delta_1]{} & H_0^{\mathrm{pcn}}(X_0). \end{array}$$

Then there is a unique natural isomorphism

$$H_1^{\mathrm{pcn}}(X_1) \xrightarrow{\phi_1} K_1(A_1),$$

fitting the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1(A_1) & \longrightarrow & K_1(A_1/A_0) & \xrightarrow{d^1_{1,0}} & K_0(A_0) & \longrightarrow & 0 \\ & & {\scriptstyle \phi_1}\uparrow{\scriptstyle \cong} & & {\scriptstyle \phi_{1,0}}\uparrow{\scriptstyle \cong} & & {\scriptstyle \phi_0}\uparrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & H_1^{\mathrm{pcn}}(X_1) & \longrightarrow & H_1^{\mathrm{pcn}}(X_1 \setminus X_0) & \xrightarrow{\partial_{1,0}} & H_0^{\mathrm{pcn}}(X_0) & \longrightarrow & 0. \end{array}$$

1C. $Codim(X) = 2$**:** We first proof that we have natural isomorphisms

$$H_*^{\mathrm{cn}}(X_l, X_m) \xrightarrow[\cong]{\phi_{l,m}} K_*(A_l/A_m) \tag{5.37}$$

for every $0 \leq m \leq l$ with $l - m = 2$ and for every manifold with corners (of any codimension). Indeed, this case can be treated very similar to the above one. Suppose l is even, the odd case is treated in the same way by exchanging K_0 by K_1 and H_0 by H_1. By comparing the long exact sequences in conormal homology we have that there exist unique natural isomorphisms $\phi^0_{l,l-2}$ and $\phi^1_{l,l-2}$ making the following diagram commutative

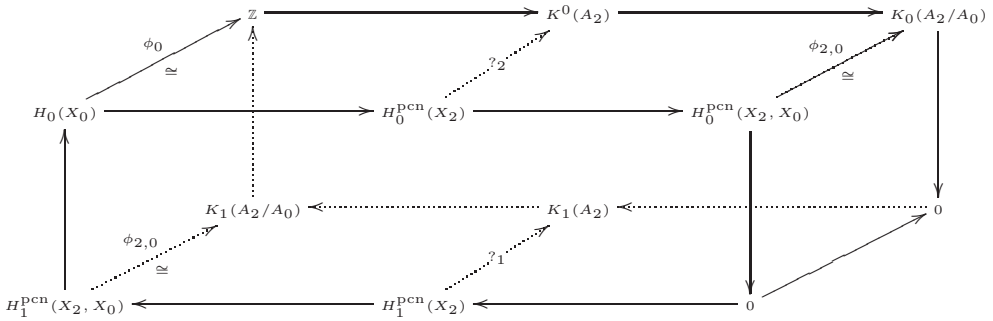
$$\begin{array}{ccccccccccc} 0 & \longrightarrow & K_0(A_l/A_{l-2}) & \longrightarrow & K_0(A_l/A_{l-1}) & \xrightarrow{d^1_{l,0}} & K_1(A_{l-1}/A_{l-2}) & \longrightarrow & K_1(A_l/A_{l-2}) & \longrightarrow & 0 \\ & & {\scriptstyle \phi^0_{l,l-2}}\uparrow{\scriptstyle \cong} & & {\scriptstyle \phi_{l,l-1}}\uparrow{\scriptstyle \cong} & & {\scriptstyle \phi_{l-1,l-2}}\uparrow{\scriptstyle \cong} & & {\scriptstyle \phi^1_{l,l-2}}\uparrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & H_0^{\mathrm{pcn}}(X_l \setminus X_{l-2}) & \longrightarrow & H_0^{\mathrm{pcn}}(X_l \setminus X_{l-1}) & \xrightarrow{\partial_{l,0}} & H_1^{\mathrm{pcn}}(X_{l-1} \setminus X_{l-2}) & \longrightarrow & H_1^{\mathrm{pcn}}(X_l \setminus X_{l-2}) & \longrightarrow & 0. \end{array}$$

since the diagram in the middle is commutative again by corollary 5.5.

Let us now pass to the case when $codim(X) = 2$. Consider the short exact sequence:

$$0 \longrightarrow A_0 \longrightarrow A_2 \longrightarrow A_2/A_0 \longrightarrow 0. \tag{5.38}$$

We compare its associated six term short exact sequence in K-theory with the one in conormal homology to get



(5.39)

where we need now to define isomorphisms $?_1$ and $?_2$. In fact if we can define morphims such that the diagrams are commutative then by a simple Five lemma argument they would be isomorphisms. The first thing to check is that

$$\begin{array}{ccc} K_1(A_2/A_0) & \xrightarrow{d_{2,0}} & K_0(A_0) \cong \mathbb{Z} \\ \phi_{2,0} \uparrow \cong & & \cong \uparrow \phi_0 \\ H_1^{\mathrm{pcn}}(X_2, X_0) & \xrightarrow[\partial_{2,0}]{} & H_0^{\mathrm{pcn}}(X_0) \end{array} \tag{5.40}$$

is commutative. Indeed, this can be seen by considering the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & A_2/A_1 & \longrightarrow & A_2/A_1 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A_0 & \longrightarrow & A_2 & \longrightarrow & A_2/A_0 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & A_0 & \longrightarrow & A_1 & \longrightarrow & A_1/A_0 & \longrightarrow & 0, \end{array} \tag{5.41}$$

applying the associated diagram between the short exact sequences that gives that the connecting morphism for the middle row, $K_1(A_2/A_0) \xrightarrow{d_{2,0}} K_0(A_0)$, is given by a (any) splitting of $K_1(A_1/A_0) \to K_1(A_2/A_0)$ (both modules are free $\mathbb{Z}$-modules by theorem 4.6) followed by the connecting morphism associated to the exact sequence on the bottom of the above diagram. By definition of $\phi_{2,0}$ in (5.37) above and by corollary 5.5 we have that these two last morphisms are compatible with the analogs in the respective conormal homologies, since the connecting morphism $\partial_{2,0}$ in conormal homology is obtained in this way as well we conclude the commutativity of (5.40). We are ready to define $?_1$ and $?_2$. For the first one, $?_1$, there is a unique isomorphism ϕ_2^1 fitting the following commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K_1(A_2) & \longrightarrow & K_1(A_2/A_0) \\ & & \phi_2^1 \uparrow \cong & & \phi_{2,0}^1 \uparrow \cong \\ 0 & \longrightarrow & H_1^{\mathrm{pcn}}(X_2) & \longrightarrow & H_1^{\mathrm{pcn}}(X_2, X_0) \end{array}$$

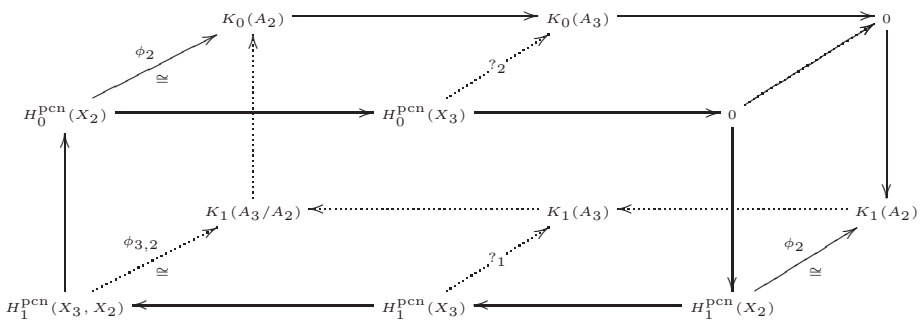
and given by restriction of $\phi_{2,0}^1$ to the image of $H_1^{\mathrm{pcn}}(X_2) \to H_1^{\mathrm{pcn}}(X_2, X_0)$. Now, for defining $?_2$ we have by theorem 5.6 an unique isomorphism ϕ_2^0 fitting the following diagram

$$\begin{array}{ccc} K_0(A_2) & \xrightarrow{\cong} & K_0(A_2/A_0) \\ \phi_2^0 \uparrow \cong & & \phi_{2,0}^0 \uparrow \cong \\ H_0^{\mathrm{pcn}}(X_2) & \xrightarrow[\cong]{} & H_0^{\mathrm{pcn}}(X_2, X_0). \end{array}$$

1D. $Codim(X) = 3$**:** Consider the short exact sequence:

$$0 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_3/A_2 \longrightarrow 0.$$

We compare its associated six term short exact sequence in K-theory with the one in conormal homology to get



(5.42)

where we need now to define isomorphisms $?_1$ and $?_2$. Again, if we can define morphims such that the diagrams are commutative then by a simple Five lemma argument they would be isomorphisms. Let us first check that the diagram

$$\begin{array}{ccc} K_1(A_3/A_2) & \xrightarrow{\partial} & K_0(A_2) \\ {\scriptstyle \phi_{3,2}}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \phi_2} \\ H_1^{\mathrm{pcn}}(X_3, X_2) & \xrightarrow[\partial]{} & H_0^{\mathrm{pcn}}(X_2) \end{array} \tag{5.43}$$

is commutative. For this consider the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_1 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_3/A_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_2/A_1 & \longrightarrow & A_3/A_1 & \longrightarrow & A_3/A_2 & \longrightarrow & 0. \end{array} \tag{5.44}$$

that implies that the connecting morphism $K_1(A_3/A_2) \xrightarrow{\partial} K_0(A_2)$ followed by the morphism $K_0(A_2) \to K_0(A_2/A_1)$ coincides with the connecting morphism $K_1(A_3/A_2) \xrightarrow{\partial} K_0(A_2/A_1)$. Now, the two latter morphisms are compatible with the analogs in conormal homology via the isomorphisms described above and the morphism $K_0(A_2) \to K_0(A_2/A_1)$ is injective (since $K_0(A_1) = 0$), hence the commutativity of diagram 5.43 above follows. From diagram (5.42), by passage to the quotient, there is unique isomorphism ϕ_3^0 (the one filling $?_2$ in the above diagram) such that

$$\begin{array}{ccccc} K_0(A_2) & \longrightarrow & K_0(A_3) & \longrightarrow & 0 \\ {\scriptstyle \phi_2^0}\uparrow{\scriptstyle \cong} & & {\scriptstyle \phi_3^0}\uparrow{\scriptstyle \cong} & & \\ H_0^{\mathrm{pcn}}(X_2) & \longrightarrow & H_0^{\mathrm{pcn}}(X_3) & \longrightarrow & 0 \end{array}$$

is commutative. Finally, for defining $?_1$, it is now enough to choose splittings for the map

$$0 \to H_1^{\mathrm{pcn}}(X_2) \to H_1^{\mathrm{pcn}}(X_3),$$

which is possible since $H_1^{\mathrm{pcn}}(X_3)$ is free (see theorem 4.6 and the remark below it) and for the map

$$K_1(A_3) \to \mathrm{im}\, j \to 0,$$

where j is the canonical morphism $j : K_1(A_3) \to K_1(A_3/A_2)$ (remember all the groups $K_*(A_p/A_{p-1})$ are torsion free).

1E. If $X = \Pi_i X_i$ is a finite product with $codim(X_i) \leq 3$ and with at least one factor of codimension at most 2: In this case the result would follow, by all the points above, if both, Periodic conormal homology and K-theory, satisfy the Künneth formula. Since the algebras $\mathcal{K}_b(X)$ are nuclear because the groupoids $\Gamma_b(X)$ are amenable we have the Künneth formula in K-theory for these kind of algebras. Now, for conormal homology we verified the Künneth formula in proposition 4.8.

1F. If $X = \Pi_i X_i$ is a finite product with $codim(X_i) \leq 3$, $\forall i$: In this case the result holds rationnaly by the same arguments as above by using propostion 4.10. □

6 Property $(\mathcal{HFP})$ and Euler conormal characters

The previous results yield a criterium for Property $(\mathcal{HFP})$ in terms of the Euler characteristic for conormal homology. To fit with the assumptions of Theorem 5.8, we consider a manifold with corners X of codimension d, which is given by the cartesian product of manifolds with corners of codimension at most 3.

Definition 6.1 (Corner characters) *Let X be a manifold with corners. We define the even conormal character of X as*

$$\chi_0(X) = dim_{\mathbb{Q}} H_0^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{6.1}$$

Similarly, we define the odd conormal character of X as

$$\chi_1(X) = dim_{\mathbb{Q}} H_1^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{6.2}$$

We can consider as well

$$\chi(X) = \chi_0(X) - \chi_1(X), \tag{6.3}$$

then we have (by the rank nullity theorem)

$$\chi(X) = 1 - \#F_1 + \#F_2 - \cdots + (-1)^d \#F_d. \tag{6.4}$$

We refer to the integer $\chi(X)$ as the Euler conormal character of X. These numbers are clearly invariant under the natural notion of isomorphism of manifolds with corners. Their computation is elementary in any concrete situation.

In particular one can rewrite the theorem above to have, for X as in the statement,

$$K_0(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{\chi_0(X)} \tag{6.5}$$

$$K_1(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{\chi_1(X)}$$

and, in terms of the corner character,

$$\chi(X) = rank(K_0(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q}) - rank(K_1(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q}). \tag{6.6}$$

In the case X is a finite product of manifolds with corners of codimension at most 2 we even have

$$K_0(\mathcal{K}_b(X)) \simeq \mathbb{Z}^{\chi_0(X)} \qquad \text{and} \qquad K_1(\mathcal{K}_b(X)) \simeq \mathbb{Z}^{\chi_1(X)} \tag{6.7}$$

and also $\chi_{\mathrm{cn}}(X) = \mathrm{rk}(K_0(\mathcal{K}_b(X))) - \mathrm{rk}(K_1(\mathcal{K}_b(X)))$.

We end with the characterization of Property $(\mathcal{HFP})$ in terms of conormal characteristics.

Theorem 6.2 *Let X be a compact connected manifold with corners of codimension greater or equal to one. If X is a finite product of manifolds with corners of codimension less or equal to three we have that*

1. *If X satisfies the Fredholm Perturbation property then the even Euler corner character of X vanishes, i.e. $\chi_0(X) = 0$.*
2. *If the even Periodic conormal homology group vanishes, i.e. $H_0^{\mathrm{pcn}}(X) = 0$ then X satisfies the stably homotopic Fredholm Perturbation property.*

3. *If $H_0^{\mathrm{pcn}}(X)$ is torsion free and if the even Euler corner character of X vanishes, i.e. $\chi_0(X) = 0$ then X satisfies the stably homotopic Fredholm Perturbation property.*

Proof :

1. Suppose $\chi_0(X) \neq 0$ then $K_0(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{\chi_0(X)}$ is not the zero group. By theorem 3.5 it is enough to prove that the rationalized analytic indicial index morphism

$$Ind_a : K^0_{top}({}^bT^*X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K_0(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is not the zero morphism. In [19] (theorem 12, 13 and proposition 7 in ref.cit.), Monthubert and Nistor construct a manifold with corners $Y$and a closed embedding of manifolds with embedded corners $X \xrightarrow{i} Y$ to obtain a commutative diagram

$$\begin{array}{ccc} K^0_{top}({}^bT^*X) \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{Ind_a} & K_0(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \Big\downarrow{\scriptstyle i!} & & \Big\downarrow{\scriptstyle \cong\, i_*} \\ K^0_{top}({}^bT^*Y) \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow[Ind_a]{\cong} & K_0(\mathcal{K}_b(Y)) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{array} \tag{6.8}$$

They call such a Y a classifying space of X. For our purposes it would be then enough to show that the morphism

$$i! : K^0_{top}({}^bT^*X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K^0_{top}({}^bT^*Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is not the zero morphism. But now we are at the topological K-theory level (with compact supports) where classic topological arguments apply to get that the morphism above is not the zero morphism. Indeed, for construct $i!$ one uses a tubular neighborhood (which exist in this setting, see for example Douady [9]), the first step is then a Thom isomorphism followed by a morphism induced by a classic extension by zero. This is summarized in proposition 5 in [19]. The conclusion follows.

2. If $H_0^{\mathrm{pcn}}(X) = 0$ then $H_0^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and the result follows from theorems 5.8 and 3.5.

3. In this case $K_0(\mathcal{K}_b(X)) \cong \mathbb{Z}^{\chi_0(X)}$ by theorem 5.8 and the arguments applied in the last two points identically apply to get the result (the results of Monthubert-Nistor cited above hold over $\mathbb{Z}$).

□

7 Appendix: more on conormal homology

We reproduce the discussion leading to the definition of the conormal differential in a slightly more general way. We keep the same notations. Let $f \in F_p$, ϵ_f an orientation of $N(f)$ and $g \in F_{p-k}$ such that $f \subset \overline{g}$. The face f is characterized in $\overline{g}$ by the vanishing of k defining functions and we denote by (g, f) the corresponding k-uple of their indices. Then the contraction $\epsilon_g := e_{(g,f)} \lrcorner \epsilon_f$ is an orientation of $N(g)$. Recall that:

$$e_J \lrcorner \cdot = e_{j_1} \lrcorner (\ldots \lrcorner (e_{j_k} \lrcorner \cdot) \ldots). \tag{7.1}$$

For any integers $0 \leq k \leq p$, we define $\delta_p^k : C_p(X) \to C_{p-k}(X)$ by

$$\delta_p^k(f \otimes \varepsilon_f) = \sum_{\substack{g \in F_{p-k}, \\ f \subset \overline{g}}} g \otimes e_{(g,f)} \lrcorner \varepsilon_f. \tag{7.2}$$

We get a homomorphism $\delta^{\mathrm{pcn}} : \mathcal{C}(X) \to \mathcal{C}(X)$ of degree 1 with respect to the $\mathbb{Z}_2$-grading by setting:

$$\delta_i^{\mathrm{pcn}} = \sum_{\substack{k \geq 0, \\ p \equiv i \bmod 2}} \delta_p^{2k+1}, \qquad i = 0, 1. \tag{7.3}$$

Proposition 7.1 *The map δ^{pcn} is a differential, that is $\delta^{\mathrm{pcn}} \circ \delta^{\mathrm{pcn}} = 0$.*

Proof : Let $f \in F_p(X)$ and ϵ be an orientation of $N(f)$. We have

$$\delta^{\mathrm{pcn}}(\delta^{\mathrm{pcn}}(f \otimes \epsilon)) = \sum_{\substack{g,h \text{ s.t. } \overline{h} \supset \overline{g} \supset f \\ (g,f),(h,g) \text{ are odd}}} \left(h \otimes e_{(h,g)} \lrcorner (e_{(g,f)} \lrcorner \varepsilon)\right). \tag{7.4}$$

Let g, h providing a term in the sum above and denote by I, J, K the uples labelling the defining functions of f, g, h respectively. Then set

$$J' = I \setminus (h, g). \tag{7.5}$$

By definition of manifolds with (embedded) corners, $H_{J'}$ is not empty and there exists a unique face $g' \in c(J')$ with $f \subset \overline{g'}$. This face $g' = \iota(g, h, f)$ satisfies the following properties:

- $f \subset \overline{g'} \subset \overline{h}$,
- $(g', f) = (h, g)$ and $(h, g') = (g, f)$ are odd,
- $\iota(g', h, f) = g$.

Finally, note that $\#(g, f) \neq \#(h, g)$, otherwise we would have $(h, f) = (h, g) + (g, f)$ even. This implies in particular that $g \neq g'$. These observations allow to reorganize the sum (7.4) as follow:

$$\delta^{\mathrm{pcn}}(\delta^{\mathrm{pcn}}(f \otimes \epsilon)) \quad = \quad \sum_{\substack{g,h \text{ s.t. } \overline{h} \supset \overline{g} \supset f \\ \#(g,f) < \#(h,g) \text{ odd}}} \left(h \otimes (e_{(h,g)} \lrcorner (e_{(g,f)} \lrcorner \varepsilon + e_{(h,g')} \lrcorner (e_{(g',f)} \lrcorner \varepsilon))\right).$$

Now

$$e_{(h,g)} \lrcorner (e_{(g,f)} \lrcorner \varepsilon) + e_{(h,g')} \lrcorner (e_{(g',f)} \lrcorner \varepsilon) \quad = \quad e_{(h,g)} \lrcorner (e_{(g,f)} \lrcorner \varepsilon) + e_{(g,f)} \lrcorner (e_{(h,g)} \lrcorner \varepsilon) = 0$$

since $\#(g, f)$ and $\#(h, g)$ are odd. □

Proposition 7.1 implies $\delta^1_{p-1} \circ \delta^1_p = 0$ for any p. Since $\delta^1_* = \delta_*$, this proves the claim of Paragraph 4.1. Moreover:

Proposition 7.2 *The identity map $(\mathcal{C}_*(X), \delta^1) \longrightarrow (\mathcal{C}_*(X), \delta)$ induces an isomorphism between the $\mathbb{Z}_2$-graded homology groups.*

Lemma 7.3 *The following equality hold for any $k \geq 0$:*

$$\delta^{2k+1} = \delta^{2k} \circ \delta^1 = \delta^1 \circ \delta^{2k} \tag{7.6}$$

Proof of the lemma : Let f be a codimension p face and ϵ an orientation of $N(f)$. Let I be the p-uple defining f. Then g is a face such that $f \subset \overline{g}$ if and only if g is a connected component of H_J for some $J \subset I$. Since the definition of $\delta(f)$ only involves faces g with $f \subset \overline{g}$, it is no restriction to remove the connected component of H_J disjoint from f for any $J \subset I$, or equivalently to assume that such H_J are connected. It follows that the faces appearing in the definition of $\delta(f)$ are in one-to-one correspondence with the uples $J \subset I$ so they can be indexed by them and eventually omitted in the sum defining $\delta^*(f)$. It follows that, ϵ_I denoting an orientation of $N(f)$,

$$\begin{aligned}
\delta^{2k} \circ \delta^1(\epsilon_I) \quad &= \quad \sum_{|J|=2k} \sum_{1 \leq i \leq N} e_J \lrcorner e_i \lrcorner \epsilon_I = \sum_{|J|=2k+1} \sum_{l=1}^{2k+1} e_{j_1} \lrcorner \cdots \lrcorner \widehat{e_{j_l}} \lrcorner \cdots \lrcorner e_{j_{2k+1}} \lrcorner e_{j_l} \lrcorner \epsilon_I \\
&= \quad \sum_{|J|=2k+1} \sum_{l=1}^{2k+1} (-1)^{l-1} e_J \lrcorner \epsilon_I = \sum_{|J|=2k+1} e_J \lrcorner \epsilon_I = \delta^{2k+1}(\epsilon_I).
\end{aligned}$$

The equality $\delta^{2k+1} = \delta^1 \circ \delta^{2k}$ is obtained in the same way. □

Proof of Proposition 7.2 : Let us set $N = \sum_{k \geq 0} \delta^{2k}$ and $h = \mathrm{Id} + N$. Using the lemma, we get:

$$\delta^{\mathrm{pcn}} = \delta^1 \circ h = h \circ \delta^1. \tag{7.7}$$

Since N is nilpotent, the map h is invertible with inverse given by the finite sum

$$h^{-1} = \sum_{j \geq 0} (-1)^j N^j.$$

This proves that $\delta^1(x) = 0$ if and only if $\delta^{\mathrm{pcn}}(x) = 0$ and that $x = \delta^1(y)$ if and only if $x = \delta^{\mathrm{pcn}}(y')$ for some y, y' as well. The proposition follows. □

The differential δ^1 is of course much simpler to handle than δ^{pcn}.

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