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STABLE GREEN RING OF THE DRINFELD DOUBLES OF THE GENERALISED TAFT ALGEBRAS (CORRECTIONS AND NEW RESULTS)

KARIN ERDMANN, EDWARD L. GREEN, NICOLE SNASHALL, AND RACHEL TAILLEFER

ABSTRACT. We return to the fusion rules for the Drinfeld double of the duals of the generalised Taft algebras that we studied in [9]. We first correct some proofs and statements in [9] that were incorrect, using stable homomorphisms. We then complete this with new results on fusion rules for the modules we had not studied in [9] and a classification of endotrivial and algebraic modules.

INTRODUCTION

Fusion rules, that is, the decomposition of tensor products of modules as a direct sum of indecomposable modules over a Hopf algebra, have been studied in several contexts, such as quantum groups or conformal field theory. In the case of quasi-triangular Hopf algebras, the tensor product of modules over the base field is commutative. Examples of quasi-triangular Hopf algebras are given by Drinfeld doubles of finite dimensional algebras. We are interested here in the Drinfeld doubles $\mathcal{D}(\Lambda_{n,d})$ of the duals $\Lambda_{n,d}$ of the extended Taft algebras over an arbitrary field k whose characteristic does not divide d, where n and d are positive integers with n a multiple of d. Such quantum doubles were originally defined by Drinfeld in order to provide solutions to the quantum Yang-Baxter equation arising from statistical mechanics. The algebra $\mathcal{D}(\Lambda_{n,d})$ has the advantage of being relatively small, and the Hopf subalgebra $\Lambda_{n,d}$ is finite-dimensional and basic, but the representations of $\mathcal{D}(\Lambda_{n,d})$ share properties with finite-dimensional representations of $U(sl_2)$ and variations. As an algebra, $\mathcal{D}(\Lambda_{n,d})$ is tame, and a parametrisation of the indecomposable modules is known. Therefore we studied direct sum decompositions for tensor products of indecomposable modules in [9]. This has also been done for the Drinfeld double of the Taft algebras, that is, the case $n = d$, in [6], using different methods.

We returned to this problem because H.-X. Chen (one of the authors of [6]) asked about the proof of [9, Theorem 4.18] (which is wrong) and the proof and statement of [9, Theorem 4.22] (they are both wrong). We are grateful to H.-X. Chen for drawing our attention to these problems. Moreover, in looking at the details, we also noticed that [9, Proposition 3.2] is incorrect, the proof of [9, Proposition 4.17] is not quite complete, and there are redundancies in our classification of $\mathcal{D}(\Lambda_{n,d})$-modules. We also realised that many of the proofs can be simplified by working over a specific block of $\mathcal{D}(\Lambda_{n,d})$.

In this paper we present a new and more homological approach to these tensor product calculations, which is based on exploiting stable module homomorphisms. This enables us to provide corrections to the proofs and statements mentioned above (except [9, Proposition 3.2], which was just a tool we used for some of our results in [9]). Using this new method, we are also able to give general formulas for the decompositions of tensor products involving the remaining modules of even length, the band modules, for which we had only given some examples in [9]. As a consequence, we now have a complete description of the stable Green ring of $\mathcal{D}(\Lambda_{n,d})$, which we give in Section 5 (Table 1 on page 17).

We also include new results, classifying endotrivial and algebraic $\mathcal{D}(\Lambda_{n,d})$-modules. If H is a finite-dimensional ribbon Hopf algebra (see for instance [5, Section 4.2.C]), then a finite-dimensional H-module M is endotrivial if there is an isomorphism $M \otimes_k M^* \cong k \oplus P$ where k is the trivial H-module and P is projective. Tensoring with an endotrivial module induces an equivalence of the stable module category, and such equivalences form a subgroup of the auto-equivalences of the stable module category. When $H = kG$ is the group algebra of a finite group G, endotrivial modules have been studied extensively. They have also been studied for finite group schemes in [3, 4]. However, we have not seen any results on endotrivial modules for other Hopf algebras. When the Hopf algebra H is our Drinfeld double $\mathcal{D}(\Lambda_{n,d})$, we show that the indecomposable endotrivial modules are precisely the syzygies of the simple modules of dimension 1 or $d-1$, see Proposition 6.3.

The concept of an algebraic module is quite natural; it was introduced as a kG-module satisfying a polynomial equation with coefficients in $\mathbb{Z}$ in the Green ring of kG, for a finite group G. We shall use the following equivalent definition: a kG-module M is called algebraic if the number of non-isomorphic indecomposable summands of the set of modules $M^{\otimes_k t}$, when $t \geqslant 1$ varies, is finite. Such modules occur in particular in the

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study of the Auslander-Reiten quiver of kG. For a study of algebraic kG-modules, see for instance [8] and the references there. Here, we replace the Hopf algebra kG with $\mathcal{D}(\Lambda_{n,d})$, and we classify the indecomposable algebraic $\mathcal{D}(\Lambda_{n,d})$-modules.

The paper is organised as follows. In Section 1, we describe the quiver and the representations of $\mathcal{D}(\Lambda_{n,d})$, in particular removing the redundancies mentioned above, and we recall the decomposition of the tensor product of two simple modules. Section 2 contains our new proof of [9, Theorem 4.18] in Theorem 2.5. Proposition 2.4 is a special case of [9, Proposition 4.17] (enough for our purposes) whose proof is now complete. Section 3 contains a corrected statement and our new proof of [9, Theorem 4.22], this is Theorem 3.4. The proof is by induction, the initial step being Proposition 3.3. In Section 4, we use these methods to determine the decomposition of the tensor product of a band module with any other indecomposable module in Propositions 4.4 and 4.6 and in Theorem 4.9. We conclude this section with a result that gives a more conceptual view of the approach we have used above. In Section 5, we combine all our results on tensor products of representations of $\mathcal{D}(\Lambda_{n,d})$ in order to describe the stable Green ring of $\mathcal{D}(\Lambda_{n,d})$. Finally, in Section 6, we present the classification of endotrivial and algebraic $\mathcal{D}(\Lambda_{n,d})$-modules.

Throughout the paper, k is an algebraically closed field and q is a fixed primitive d-th root of unity in k (in particular, the characteristic of k does not divide d). All modules are left modules. The cyclic group of order n is denoted by $\mathbb{Z}_n$ and $\otimes$ denotes the tensor product over k.

We refer to [9] for all the background on the representation theory of $\mathcal{D}(\Lambda_{n,d})$, where n and d are integers such that d divides n. However, we recall (mainly in Section 1) the definitions and results from [9] that are required for the understanding of this paper, so that the reader does not need to refer to our original paper.

1. Parametrization of modules over $\mathcal{D}(\Lambda_{n,d})$

In this section, we recall briefly a description of the algebras $\Lambda_{n,d}$, $\Lambda_{n,d}^{*cop}$ and $\mathcal{D}(\Lambda_{n,d})$, as well as the isomorphism classes of representations of $\mathcal{D}(\Lambda_{n,d})$. We refer to [9, Sections 2 and 3] for more details.

***Notation* 1.1.** As in [9, Notation 2.5 and Definition 2.11], we denote by $\langle r\rangle$ the residue of an integer r modulo d taken in the set $\{1,2,\dots,d\}$ and, for any $u\in\mathbb{Z}_n$, we define a permutation σ_u of $\mathbb{Z}_n$ by

$$\sigma_u(j) = d + j - \langle 2j+u-1\rangle$$

(recall that d divides n). If $2j+u-1$ is not divisible by d then the orbit of j under σ_u has size $2\frac{n}{d}$ and moreover we have $\sigma_u^{2t}(i) = i+td$ and $\sigma_u(i)^{2t+1}(i) = \sigma_u(i)+td$ in $\mathbb{Z}_n$.

1.1. **The original algebras.** The algebra $\Lambda_{n,d}$ is described by quiver and relations. Its quiver is the cyclic quiver with vertices $e_1,\dots,e_{n-1}$ and arrows $a_0,\dots,a_{n-1}$, where the indices are viewed in $\mathbb{Z}_n$ and each arrow a_i goes from e_i to e_{i+1}. The ideal of relations of $\Lambda_{n,d}$ is generated by the paths of length d. We also denote by $\gamma_i^m = a_{i+m-1}\cdots a_{i+1}a_i$ the path of length m that starts at e_i.

Since we assume that d divides n, this algebra is a Hopf algebra by [7]. We shall only need the antipode here, which is determined by

$$S(e_i) = e_{-i} \text{ and } S(a_i) = -q^{i+1}a_{-i-1} \text{ for all } i\in\mathbb{Z}_n.$$

The co-opposite of the dual Hopf algebra, $\Lambda_{n,d}^{*cop}$, is the extended Taft algebra, and it is presented by generators and relations:

$$\Lambda_{n,d}^{*cop} = \langle G, X \mid G^n = 1,\ X^d = 0,\ GX = q^{-1}XG\rangle.$$

Its antipode is determined by $S(G) = G^{-1}$ and $S(X) = -XG^{-1}$.

These algebras are both Hopf subalgebras of the Drinfeld double $\mathcal{D}(\Lambda_{n,d})$, that is equal to $\Lambda_{n,d}\otimes\Lambda_{n,d}^{*cop}$ as a vector space, and has a basis given by the set of $G^iX^j\gamma_\ell^m$ with i,ℓ in $\mathbb{Z}_n$ and $0\leqslant j,m\leqslant d-1$. In this paper we shall not need the relations between the generators.

1.2. **$\mathcal{D}(\Lambda_{n,d})$-modules of odd length, projective $\mathcal{D}(\Lambda_{n,d})$-modules and blocks of $\mathcal{D}(\Lambda_{n,d})$.** The simple $\mathcal{D}(\Lambda_{n,d})$-modules are labelled $L(u,i)$ for $(u,i)\in\mathbb{Z}_n^2$. The description below is taken from [9, Section 2, mainly Propositions 2.17 and 2.21].

The module $L(u,i)$ is projective if and only if $2i+u-1$ is divisible by d, in which case $\dim L(u,i) = d$. When $L(u,i)$ is not projective, then $\dim L(u,i) = d - \langle 2i+u-1\rangle = \sigma_u(i) - i$ and $L(u,i)$ contains two distinguished vectors $\tilde{H}_{u,i}$ and $\tilde{F}_{u,i}$, with the following properties:

(a) $\tilde{H}_{u,i}$ spans the kernel of the action of X on $L(u,i)$ as a vector space and $e_j\tilde{H}_{u,i} = \delta_{ij}\tilde{H}_{u,i}$,

(b) $e_j\tilde{F}_{u,i} = \delta_{j,\sigma_u(i)-1}\tilde{F}_{u,i}$ and the element $\tilde{F}_{u,i}$ is annihilated by all the arrows in the quiver of $\Lambda_{n,d}$,

where δ is the Kronecker symbol. The module $L(u,i)$ has basis $\{X^t\tilde{F}_{u,i} : 0\leqslant t < \dim L(u,i) = N\}$ and $X^{N-1}\tilde{F}_{u,i}$ is a non-zero scalar multiple of $\tilde{H}_{u,i}$. Moreover, the action of G on these basis elements is given by $GX^t\tilde{F}_{u,i} = q^{-i-t}\tilde{F}_{u,i}$.

In [9, Proposition 2.21], we characterised the simple modules as follows.

Proposition 1.2. *Let S be a simple module. Set $E_u = \frac{1}{n}\sum_{i,j\in\mathbb{Z}_n} q^{-i(u+j)}G^i e_j$ for $u \in \mathbb{Z}_n$. Then S is isomorphic to $L(u,i)$ if and only if the three following properties hold:*

(a) $\dim S = \dim L(u,i)$.
(b) *E_u acts as identity on S, and E_v acts as zero on S if $v \neq u$.*
(c) *Let Y be the generator of S which is in the kernel of the action of X (this is well-defined up to a non-zero scalar and corresponds to $\tilde{H}_{u,i}$). Then the vertex e_i acts as identity on Y, and the other vertices act as zero.*

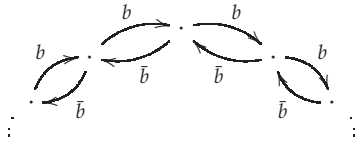
The projective cover $P(u,i)$ of $L(u,i)$ has four composition factors. The socle and the top are isomorphic to $L(u,i)$, and

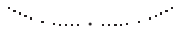
$$\operatorname{rad} P(u,i)/\operatorname{soc} P(u,i) \cong L(u,\sigma_u(i)) \oplus L(u,\sigma_u^{-1}(i)).$$

The **indecomposable modules of odd length** are precisely the syzygies of the non-projective simple modules, that is, the $\Omega^m(L(u,i))$ for $m \in \mathbb{Z}$ and $(u,i) \in \mathbb{Z}_n^2$ where d does not divide $2i+u-1$.

It follows from the structure of the projective modules that the simple modules in the block of $L(u,i)$ are precisely the simple modules $L(u,\sigma_u^t(i))$ for all t, and there are $2\frac{n}{d}$ of them. Theorem 2.26 of [9] completely describes the basic algebra of a non-simple block. Each block is symmetric and special biserial with radical cube zero.

More precisely, as an algebra, $\mathcal{D}(\Lambda_{n,d})$ is the direct sum of $\frac{n^2}{d}$ simple blocks and of $\frac{n(d-1)}{2}$ blocks $\mathbb{B}_{u,i}$ for $(u,i) \in \mathbb{Z}_n^2$ such that $2i+u-1$ is not a multiple of d. The quiver of $\mathbb{B}_{u,i}$ is





with $\frac{2n}{d}$ vertices and $\frac{4n}{d}$ arrows. The relations on this quiver are bb, $\bar{b}\bar{b}$ and $b\bar{b}-\bar{b}b$ (there are $\frac{6n}{d}$ relations on each of these quivers). The vertices in this quiver correspond to the simple modules $L(u,i)$, $L(u,\sigma_u(i))$, $L(u,\sigma_u^2(i))$, $\ldots$, $L(u,\sigma_u^{\frac{2n}{d}-1}(i))$. Hence $\mathbb{B}_{u,i} = \mathbb{B}_{v,j}$ if and only if $u=v$ and $j=\sigma_u^t(i)$ for some $t \in \mathbb{Z}$.

Moreover, the arrows b and $\bar{b}$ are described as follows. For each p, there are basis elements in $L(u,\sigma^p(i))$ that, following the notation in [9, Propostion 2.17], we denote by $\tilde{D}_{u,\sigma_u^{p+1}(i)}$ and $F_{u,\sigma_u^{p-1}(i)}$, such that:

- the action of b_p on $L(u,\sigma^p(i))$ is given by multiplication by $\gamma_{\sigma_u^p(i)}^{\dim L(u,\sigma_u^p(i))}$ and $\gamma_{\sigma_u^p(i)}\tilde{D}_{u,\sigma_u^{p+1}(i)}$ is a non-zero scalar multiple of $\tilde{H}_{u,\sigma_u^{p+1}(i)}$,
- the action of $\bar{b}_{p-1}$ on $L(u,\sigma^p(i))$ is given by multiplication by $X^{\dim L(u,\sigma_u^p(i))}$ and $X^{\dim L(u,\sigma_u^p(i))}F_{u,\sigma_u^{p-1}(i)}$ is equal to $\tilde{F}_{u,\sigma_u^{p-1}(i)}$.

1.3. **$\mathcal{D}(\Lambda_{n,d})$-modules of even length.** We described all the non-projective indecomposable representations of $\mathcal{D}(\Lambda_{n,d})$ in [9, Section 3 and Appendix A]. These are

(a) The modules of odd length described above.
(b) The string modules of length 2ℓ, denoted by $M_{2\ell}^+(u,i)$ and $M_{2\ell}^-(u,i)$, for any positive integer ℓ and $(u,i) \in \mathbb{Z}_n^2$ with $2i+u-1 \not\equiv 0 \pmod d$.
(c) The band modules $C_\lambda^\ell(u,i)$ of length $2\ell\frac{n}{d}$ for $\lambda \in k\setminus\{0\}$, any positive integer ℓ and $(u,i) \in \mathbb{Z}_n^2$ with $2i+u-1 \not\equiv 0 \pmod d$, taking i modulo d (that is, one for each orbit of the square of σ_u).

We now describe the modules of even length in more detail for future use. Fix a block $\mathbb{B}_{u,i}$.

1.3.1. *String modules of even length.* For each $0 \leqslant p \leqslant \frac{2n}{d} - 1$ and for each $\ell \geqslant 1$, there are two indecomposable modules of length 2ℓ which we call $M^{\pm}_{2\ell}(u, \sigma_u^p(i))$:

- The module $M^{+}_{2\ell}(u,i)$ has top composition factors $L(u,i)$, $L(u,\sigma_u^2(i))$, $\dots$, $L(u,\sigma_u^{2(\ell-1)}(i))$ and socle composition factors $L(u,\sigma_u(i))$, $L(u,\sigma_u^3(i))$, $\dots$, $L(u,\sigma_u^{2(\ell-1)+1}(i))$:

$$\begin{array}{ccccccc} L(u,i) & & L(u,\sigma_u^2(i)) & & L(u,\sigma_u^{2(\ell-1)}(i)) & & \\ & \searrow & & \swarrow \quad \searrow & & \swarrow & \\ & L(u,\sigma_u(i)) & & \cdots & & L(u,\sigma_u^{2(\ell-1)+1}(i)) \end{array}$$

  The lines joining the simple modules are given by multiplication by the appropriate b-arrow or $\bar{b}$-arrow (in the case $n = d$, when there is an ambiguity, the first line is multiplication by $\gamma^{\dim L(u,i)}$, the next one is multiplication by a non-zero scalar multiple of $X^{d-\dim L(u,i)}$, and so on, up to non-zero scalars).
- The module $M^{-}_{2\ell}(u,i)$ has top composition factors $L(u,i)$, $L(u,\sigma_u^{-2}(i))$, $\dots$, $L(u,\sigma_u^{-2(\ell-1)}(i))$ and socle composition factors $L(u,\sigma_u^{-1}(i))$, $L(u,\sigma_u^{-3}(i))$, $\dots$, $L(u,\sigma_u^{-2(\ell-1)+1}(i))$:

$$\begin{array}{ccccccc} & & L(u,\sigma_u^{-2(\ell-1)}(i)) & & L(u,\sigma_u^{-2}(i)) & & L(u,i) \\ & \swarrow & & \searrow \quad \swarrow & & \searrow & \\ L(u,\sigma_u^{-2(\ell-1)+1}(i)) & & & \cdots & & L(u,\sigma_u^{-1}(i)) & \end{array}$$

  As for the other string modules, the lines represent multiplication by an appropriate b or $\bar{b}$ arrow, and in the case $n = d$ the first one from the left is multiplication by a power of X and so on.

In both cases, indices are taken modulo $\frac{2n}{d}$.

These modules are periodic of period $\frac{2n}{d}$. Moreover, the Auslander-Reiten sequences of the string modules $M^{\pm}_{2\ell}(u,i)$ are given by

$$\begin{aligned} \mathcal{A}(M^{+}_{2\ell}(u,i)) : &\qquad 0 \to M^{+}_{2\ell}(u,i-d) \to M^{+}_{2\ell+2}(u,i-d) \oplus M^{+}_{2\ell-2}(u,i) \to M^{+}_{2\ell}(u,i) \to 0, \\ \mathcal{A}(M^{-}_{2\ell}(u,i)) : &\qquad 0 \to M^{-}_{2\ell}(u,i+d) \to M^{-}_{2\ell+2}(u,i+d) \oplus M^{-}_{2\ell-2}(u,i) \to M^{-}_{2\ell}(u,i) \to 0 \end{aligned}$$

(where $M^{\pm}_{0}(u,i) = 0$).

1.3.2. *Band modules.* For each $\lambda \neq 0$ in k, and for each $\ell \geqslant 1$, there is an indecomposable module of length $\frac{2n}{d}\ell$, which we denote by $C^{\ell}_{\lambda}(u,i)$. It is defined as follows. Denote by ϵ_p, for $p \in \mathbb{Z}_{2n/d}$, the vertices in the quiver of $\mathbb{B}_{u,i}$, with ϵ_0 corresponding to $L(u,i)$. The arrow b_p goes from ϵ_p to ϵ_{p+1} and the arrow $\bar{b}_p$ goes from ϵ_{p+1} to ϵ_p. Let V be an ℓ-dimensional vector space. Then $C^{\ell}_{\lambda}(u,i)$ has underlying space $C = \bigoplus_{p=0}^{\frac{2n}{d}-1} C_p$ with $C_p = V$ for all p. The action of the idempotents ϵ_p is such that $\epsilon_p C = C_p$. The action of the arrows $\bar{b}_{2p}$ and b_{2p+1} is zero. The action of the arrows $\bar{b}_{2p+1}$ is the identity of V. The action of the arrows b_{2p} with $p \neq 0$ is also the identity. Finally, the action of b_0 is given by the indecomposable Jordan matrix $J_\ell(\lambda)$.

It is periodic of period 2 and its Auslander-Reiten sequence is given by

$$\mathcal{A}(C^{\ell}_{\lambda}(u,i)) : \qquad 0 \to C^{\ell}_{\lambda}(u,i-d) \to C^{\ell+1}_{\lambda}(u,i) \oplus C^{\ell-1}_{\lambda}(u,i) \to C^{\ell}_{\lambda}(u,i) \to 0$$

(where $C^{0}_{\lambda}(u,i) = 0$).

Note that $\operatorname{soc}(C) = \operatorname{rad}(C) = \bigoplus_p \epsilon_{2p+1} C$ and that $C/\operatorname{rad}(C) = \bigoplus_p \epsilon_{2p} C$.

***Remark* 1.3.** In [9], we denoted $C^{\ell}_{\lambda}(u,i)$ by $C^{\ell+}_{\lambda}(u,i)$, and we had more band modules. First note that $C^{\ell}_{\lambda}(u,i)$ is isomorphic to $C^{\ell}_{\lambda}(u,\sigma_u^2(i))$, so that in (c) above we have removed some repetitions that occurred in [9].

In addition, in [9] we also had the modules $C^{\ell-}_{\lambda}(u,i)$, defined in a similar way by interchanging b's and $\bar{b}$'s. However, we should have noted that for each $\lambda \in k \setminus \{0\}$, there exists $\mu \in k \setminus \{0\}$ and $(v,j) \in \mathbb{Z}_n^2$ such that $C^{\ell-}_{\lambda}(u,i) \cong C^{\ell}_{\mu}(v,j)$.

Indeed, by definition of the module $C^{\ell-}_{\lambda}(u,i)$, the action of $\bar{b}_0$ is given by $J_\ell(\lambda)$, the action of $\bar{b}_i$ for i even, $i \neq 0$, is given by id, the action of b_i for i odd is given by id and the action of the other arrows is given by 0. Changing bases of the vector spaces C^{-}_{p}, we can ensure that the action of b_{-1} is given by $J_\ell\left(\frac{1}{\lambda}\right)$, the action of $\bar{b}_0$ is given by id and the rest is unchanged, thus obtaining the module $C^{\ell}_{\frac{1}{\lambda}}(u,\sigma_u^{-1}(i))$. Hence $C^{\ell-}_{\lambda}(u,i) \cong C^{\ell}_{\frac{1}{\lambda}}(u,\sigma_u^{-1}(i))$.

Therefore we need only consider one family of band modules, which we denote by $C^{\ell}_{\lambda}(u,i)$ for $\lambda \in k \setminus \{0\}$, and we need only one of these for each orbit of σ_u^2 in $\mathbb{Z}_n$.

However, it should be noted that this parameter μ such that $C^{\ell-}_{\lambda}(u,i) \cong C^{\ell}_{\mu}(v,j)$ is not well defined, unless we have fixed the block $\mathbb{B}_{u,i}$ (including which vertex is labelled ϵ_0): using the definition in [9, Section A.2.2], where the representations of the basic algebra of a given block are described, the module $C^{\ell-}_{\lambda}(u,i)$ is defined

by considering that the block $\mathbb{B}_{u,i}$ is equal to $\mathbb{B}_{u,\sigma_u(i)}$ with a shift in the indices of the vertices and arrows by 1; here we get immediately that $C_\lambda^{\ell-}(u,i) = C_\lambda^\ell(u,i)$.

As we mentioned above, there is a difficulty in differentiating modules of length $2\frac{n}{d}$, since $M^+_{2\frac{n}{d}}(u,i)$, $M^-_{2\frac{n}{d}}(u,i)$ and $C^1_\lambda(u,i)$ are modules of length $2\frac{n}{d}$ with the same top and the same socle. The following property allows us to distinguish them. Recall that $\Lambda_{n,d}$ and $\Lambda_{n,d}^{*cop}$ are subalgebras of $\mathcal{D}(\Lambda_{n,d})$, so that any $\mathcal{D}(\Lambda_{n,d})$-module can be viewed as a $\Lambda_{n,d}$-module and as a $\Lambda_{n,d}^{*cop}$-module.

Property 1.4. *Assume that d does not divide $2i+u-1$. Then*

- *the modules $M^+_{2\ell}(u,i)$ are projective as $\Lambda_{n,d}$-modules but not as $\Lambda_{n,d}^{*cop}$-modules,*
- *the modules $M^-_{2\ell}(u,i)$ are projective as $\Lambda_{n,d}^{*cop}$-modules but not as $\Lambda_{n,d}$-modules,*
- *the modules $C^\ell_\lambda(u,i)$ are projective both as $\Lambda_{n,d}$-modules and as $\Lambda_{n,d}^{*cop}$-modules.*

Proof. We use the notation from the last part of Subsection 1.2.

It follows from [7] and [10] that the indecomposable projective modules over $\Lambda_{n,d}$ or $\Lambda_{n,d}^{*cop}$ are precisely the indecomposable modules of dimension d. Therefore $M^+_2(u,i)$, which is equal to $\Lambda_{n,d}\tilde{D}_{u,\sigma_u(i)}$ as a $\Lambda_{n,d}$-module, is indecomposable of dimension d, hence projective. Moreover, as a $\Lambda_{n,d}$-module, $M^+_{2\ell}(u,i) \cong \bigoplus_{t=0}^{d-1} M^+_2(u,i+td)$ is also projective.

Similarly, the $\Lambda_{n,d}^{*cop}$-module $M^-_2(u,i) = \Lambda_{n,d}^{*cop} F_{u,\sigma_u^{-1}(j)}$ is projective, and $M^-_{2\ell}(u,i) \cong \bigoplus_{t=0}^{\ell-1} M^-_2(u,i-td)$ is a projective $\Lambda_{n,d}^{*cop}$-module.

As a $\Lambda_{n,d}^{*cop}$-module, $M^+_{2\ell}(u,i)$ decomposes as $L(u,i) \oplus \bigoplus_{t=1}^{\ell-1} M^-_2(u,i+td) \oplus L(u,\sigma_u(i)+(\ell-1)d)$, therefore it has a summand $L(u,i)$ whose dimension is strictly less than d, hence that is not projective. Therefore $M^+_{2\ell}(u,i)$ is not projective.

Similarly, $M^-_{2\ell}(u,i)$ is not projective as a $\Lambda_{n,d}$-module.

Finally, as a $\Lambda_{n,d}$-module, $C^\ell_\lambda(u,i) \cong \bigoplus_{t=0}^{\frac{n}{d}-1} M^+_2(u,i+td)^\ell$ is projective, and as a $\Lambda_{n,d}^{*cop}$-module, $C^\ell_\lambda(u,i) \cong \bigoplus_{t=0}^{\frac{n}{d}-1} M^-_2(u,i-td)^\ell$ is projective. □

***Remark* 1.5.** It is clear that the $\mathcal{D}(\Lambda_{n,d})$-modules $P(u,i)$ are projective both as $\Lambda_{n,d}$-modules and as $\Lambda_{n,d}^{*cop}$-modules. Moreover, the non-projective simple $\mathcal{D}(\Lambda_{n,d})$-modules $L(u,i)$ have dimension at most $d-1$ and therefore they are not projective as $\Lambda_{n,d}$-modules or as $\Lambda_{n,d}^{*cop}$-modules.

In this paper, we consider tensor products of $\mathcal{D}(\Lambda_{n,d})$-modules over the base field. The tensor products of $\Lambda_{n,d}$-modules and of $\Lambda_{n,d}^{*cop}$-modules have been studied in [7] and [10]. It follows in particular from their results that if M and N are two indecomposable $\Lambda_{n,d}$-modules (respectively $\Lambda_{n,d}^{*cop}$-modules), then $M \otimes N$ is projective if and only if M or N is projective.

Definition 1.6. *We shall say that two string modules $M^+_{2\ell}(u,i)$ and $M^+_{2t}(u,j)$ (respectively two string modules $M^-_{2\ell}(u,i)$ and $M^-_{2t}(u,j)$, respectively two band modules $C^\ell_\lambda(u,i)$ and $C^t_\mu(u,j)$) have the* same parity *if $i \equiv j \pmod d$ and that they have* different parities *if $j \equiv \sigma_u(i) \pmod d$.*

This definition is consistent with [9, Definition A.3].

Definition 1.7. *We shall write* $\mathrm{core}(M)$ *for the direct sum of the non-projective indecomposable summands in M.*

In [9, Theorem 4.1], we determined the decomposition of the tensor product of two simple modules. Here we shall need the core of this module.

Proposition 1.8. *(cf.* [9, Theorem 4.1]*) Let $L(u,i)$ and $L(v,j)$ be two non-projective simple $\mathcal{D}(\Lambda_{n,d})$-modules. Then*

$$\mathrm{core}(L(u,i)\otimes L(v,j)) \cong \bigoplus_{\theta\in\mathfrak{I}} L(u+v,i+j+\theta)$$

where $\mathfrak{I} = \begin{cases} \{\theta\,;\, 0 \leqslant \theta \leqslant \min(\dim L(u,i),\dim L(v,j))-1\} \text{ if } \dim L(u,i)+\dim L(v,j) \leqslant d \\ \{\theta\,;\, \dim L(u,i)+\dim L(v,j)-d \leqslant \theta \leqslant \min(\dim L(u,i),\dim L(v,j))-1\} \text{ otherwise.} \end{cases}$

Moreover, if $\dim L(u,i)+\dim L(v,j) \leqslant d$, we have $L(u,i)\otimes L(v,j) = \mathrm{core}(L(u,i)\otimes L(v,j))$.

Proof. Set $N = \dim L(u,i)$ and $N' = \dim L(v,j)$. Without loss of generality, we can assume that $N \leqslant N'$. We proved in [9, Theorem 4.1] that

$$L(u,i)\otimes L(v,j) \cong \begin{cases} \bigoplus_{\theta=0}^{N-1} L(u+v,i+j+\theta) & \text{if } N+N' \leqslant d+1 \\ \bigoplus_{\theta=N+N'-d}^{N-1} L(u+v,i+j+\theta) \oplus \text{ projective} & \text{if } N+N' \geqslant d+1 \end{cases}$$

so we need only determine which of the simple modules are projective. Note that $2i+u-1 \equiv -N \pmod{d}$ and $2j+v-1 \equiv -N' \pmod{d}$, therefore $2(i+j+\theta)+(u+v)-1 \equiv -N-N'+2\theta+1 \pmod{d}$. Moreover, since $\theta \leqslant N-1 \leqslant N'-1$, we have $-N-N'+2\theta+1 \leqslant -1$.

If $N+N' \leqslant d$ we have $-N-N'+2\theta+1 \geqslant -d+1$, and if $N+N' > d+1$ we have $-N-N'+2\theta+1 \geqslant -N-N'+2(N+N'-d)+1 = N+N'-2d+1 > d+1-2d+1 = 2-d$, therefore in both cases $2(i+j+\theta)+(u+v)-1 \not\equiv 0 \pmod{d}$ and $L(u+v, i+j+\theta)$ is not projective.

If $N+N' = d+1$, we have $-N-N'+2\theta+1 \geqslant 2\theta-d \equiv 2\theta \pmod{d}$ with $0 \leqslant 2\theta \leqslant 2(N-1) \leqslant N+N'-2 = d-1$, therefore $2(i+j+\theta)+(u+v)-1 \equiv 0 \pmod{d}$ if and only if $\theta = 0$.

Finally, the only projective that can occur is $L(u+v, i+j)$ when $N+N' = d+1$. The result then follows. □

2. Decomposition of the tensor product of a simple module with a string module of even length

The stable Green ring $r_{st}(H)$ of a finite dimensional Hopf algebra H is the ring whose $\mathbb{Z}$-basis is the set of isomorphism classes of indecomposable modules in the stable category H-$\underline{\text{mod}}$, that is, the non-projective indecomposable H-modules, and whose addition and multiplication are induced respectively by the direct sum and the tensor product.

The aim of this section is to determine the tensor product of a string module of even length with a simple module up to projectives, from which we can deduce the tensor product of a string module of even length with a string module of odd length in the stable Green ring $r_{st}(\mathcal{D}(\Lambda_{n,d}))$. This will provide a correction to the proof of [9, Theorem 4.18].

For the proof, we will use a general result involving splitting trace modules. We recall that a module M is a *splitting trace module* if the trivial module $L(0,0)$ is a direct summand in $\mathrm{End}_k(M)$. Moreover, for any ribbon Hopf algebra such as $\mathcal{D}(\Lambda_{n,d})$, there are isomorphisms $\mathrm{End}_k(M) \cong M^* \otimes M \cong M \otimes M^*$ of $\mathcal{D}(\Lambda_{n,d})$-modules for any module M (where M^* is the k-dual of M), see for instance [5, Section 4.2.C].

The **indecomposable splitting trace modules for** $\mathcal{D}(\Lambda_{n,d})$ were given in [9, Subsection 3.3]: they are precisely the non-projective indecomposable modules of odd length.

Lemma 2.1. *Let C be an indecomposable non-projective $\mathcal{D}(\Lambda_{n,d})$-module and let $\mathcal{A}(C)$ be its Auslander-Reiten sequence. For any $\mathcal{D}(\Lambda_{n,d})$-module M, the following are equivalent.*

(i) C is a direct summand in $\mathrm{End}_k(M) \otimes C$.
(ii) C is a direct summand in $M^* \otimes M \otimes C$.
(iii) The sequence $\mathcal{A}(C) \otimes M$ does not split.
(iv) If $0 \to A \to B \to C \to 0$ is any non-split exact sequence of $\mathcal{D}(\Lambda_{n,d})$-modules, then the exact sequence $0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$ does not split.

If moreover M is a splitting trace module, then all these properties hold.

Proof. The equivalence of the first two statements follows from the isomorphism $\mathrm{End}_k(M) \cong M^* \otimes M$ of $\mathcal{D}(\Lambda_{n,d})$-modules. The equivalence between statements (ii), (iii) and (iv) is proved in the same way as [1, Proposition 2.3]. Finally, if M is a splitting trace module, then $L(0,0)$ is a summand in $\mathrm{End}_k(M)$ so that $C \cong L(0,0) \otimes C$ is a direct summand in $\mathrm{End}_k(M) \otimes C$ and (i) holds. □

We start with the special cases of the tensor product of $M_2^{\pm}(0,td)$ with a simple module in Proposition 2.2 and of the tensor product of $M_{2\ell}^{\pm}(0,td)$ with a simple module in Proposition 2.4, from which we deduce the general case in Theorem 2.5.

Proposition 2.2. *(cf. [9, Proposition 4.17]) For all $t \in \mathbb{Z}$, we have $\mathrm{core}(M_2^+(0,td) \otimes L(v,j)) \cong M_2^+(v, j+td)$ and $\mathrm{core}(M_2^-(0,td) \otimes L(v,j)) \cong M_2^-(v, j+td)$.*

Proof. Set $N = \dim L(v,j)$. We tensor the non-split exact sequence $0 \to L(0,1+td) \to M_2^+(0,td) \to L(0,td) \to 0$ by $L(v,j)$. Using Proposition 1.8 as well as $\dim L(0,td) = 1$ and $\dim L(0,1+td) = d-1$ (so that $\#\mathfrak{I} = 1$), we obtain an exact sequence $0 \to L(v, j+td+N) \oplus P \to M_2^+(0,td) \otimes L(v,j) \to L(v, j+td) \to 0$ with P a projective module. Since P is injective, we have $M_2^+(0,td) \otimes L(v,j) \cong U \oplus P$ for some module U and we obtain an exact sequence $0 \to L(v, j+td+N) \to U \to L(v, j+td) \to 0$. The sequence cannot split by Lemma 2.1 because $L(u,j)$ is a splitting trace module, so U is an indecomposable module of length 2 with $\mathrm{top}\, U = L(v, j+td)$ and $\mathrm{soc}\, U = L(v, j+td+N) = L(v, \sigma_v(j+td))$. Moreover, it follows from Property 1.4 and Remark 1.5 that $M_2^+(0,td) \otimes L(v,j)$, and hence U, is projective as a $\Lambda_{n,d}$-module but not as a $\Lambda_{n,d}^{*cop}$-module, therefore we must have $U = M_2^+(v, j+td)$.

The proof of the second part is similar. □

In the next lemma as well as in Proposition 3.3, we shall use properties of homomorphisms between string modules of even length over a given block; these properties can easily be seen by working over the basic algebra of the block.

Lemma 2.3. *Let t and ℓ be integers with $1 \leqslant t \leqslant \ell$. Then the spaces $\mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(M^+_{2\ell}(u,i+d), M^+_{2t}(u,i))$ and $\mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(M^-_{2\ell}(u,i-d), M^-_{2t}(u,i))$ have dimension*

$$\#\{0 \leqslant y \leqslant t-1 \,|\, y \equiv 0 \ (\mathrm{mod}\ \tfrac{n}{d})\} - \#\{0 \leqslant y \leqslant t-1 \,|\, y \equiv \ell \ (\mathrm{mod}\ \tfrac{n}{d})\} + \#\{0 \leqslant y \leqslant t-1 \,|\, y \equiv t-\ell-1 \ (\mathrm{mod}\ \tfrac{n}{d})\}.$$

In particular, $\mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(M^+_{2\ell}(u,i+d), M^+_2(u,i))$ and $\mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(M^-_{2\ell}(u,i-d), M^-_2(u,i))$ have dimension 1. A basis of $\mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(M^+_{2\ell}(u,i+d), M^+_2(u,i))$ (respectively $\mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(M^-_{2\ell}(u,i-d), M^-_2(u,i))$) is given by the equivalence class of the exact sequence

$$0 \to M^+_2(u,i) \to M^+_{2\ell+2}(u,i) \to M^+_{2\ell}(u,i+d) \to 0$$
$$(\textit{respectively} \quad 0 \to M^-_2(u,i) \to M^-_{2\ell+2}(u,i) \to M^-_{2\ell}(u,i-d) \to 0).$$

Proof. We prove the first statement, the proof of the second is similar.

Applying $\mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(-, M^+_{2t}(u,i))$ to the exact sequence

$$0 \to \Omega(M^+_{2\ell}(u,i+d)) \cong M^+_{2\ell}(u,\sigma_u(i)) \to Q := \bigoplus_{x=1}^{\ell} P(u,i+xd) \to M^+_{2\ell}(u,i+d) \to 0$$

gives an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(M^+_{2\ell}(u,i+d), M^+_{2t}(u,i)) \to \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(Q, M^+_{2t}(u,i)) \to \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(M^+_{2\ell}(u,\sigma_u(i)), M^+_{2t}(u,i))$$
$$\to \mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(M^+_{2\ell}(u,i+d), M^+_{2t}(u,i)) \to 0$$

so that

$$\begin{aligned}\dim \mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(M^+_{2\ell}(u,i+d), M^+_{2t}(u,i)) &= \dim \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(M^+_{2\ell}(u,i+d), M^+_{2t}(u,i)) \\ &\quad - \dim \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(Q, M^+_{2t}(u,i)) \\ &\quad + \dim \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(M^+_{2\ell}(u,\sigma_u(i)), M^+_{2t}(u,i)).\end{aligned}$$

We determine each of these dimensions.

We have $\dim \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(Q, M^+_{2t}(u,i)) = \sum_{x=1}^{\ell} \#\{y\,;\, 0 \leqslant y \leqslant t-1,\ y \equiv x \ (\mathrm{mod}\ \frac{n}{d})\}$.

Since $M^+_{2\ell}(u,\sigma_u(i))$ and $M^+_{2t}(u,i)$ have different parities, any non-zero map in $\mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(M^+_{2\ell}(u,\sigma_u(i)), M^+_{2t}(u,i))$ must send the top of $M^+_{2\ell}(u,\sigma_u(i))$ into the socle of $M^+_{2t}(u,i)$. Therefore

$$\begin{aligned}\dim \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(M^+_{2\ell}(u,\sigma_u(i)), M^+_{2t}(u,i)) &= \sum_{x=0}^{\ell-1} \#\{y\,;\, 0 \leqslant y \leqslant t-1,\ y \equiv x \ (\mathrm{mod}\ \frac{n}{d})\} \\ &= \dim \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(Q, M^+_{2t}(u,i)) \\ &\quad + \#\{y\,;\, 0 \leqslant y \leqslant t-1,\ y \equiv 0 \ (\mathrm{mod}\ \frac{n}{d})\} \\ &\quad - \#\{y\,;\, 0 \leqslant y \leqslant t-1,\ y \equiv \ell \ (\mathrm{mod}\ \frac{n}{d})\}.\end{aligned}$$

Since $M^+_{2\ell}(u,i+d)$ and $M^+_{2t}(u,i)$ have the same parity, the image of any non-zero map in $\mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(M^+_{2\ell}(u,i+d), M^+_{2t}(u,i))$ is a quotient of $M^+_{2\ell}(u,i+d)$ and a string submodule of $M^+_{2t}(u,i)$ that starts at $L(u,i)$. Therefore, since $\ell \geqslant t$,

$$\begin{aligned}\dim \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(M^+_{2\ell}(u,i+d), M^+_{2t}(u,i)) &= \#\{x\,;\, \ell-t+1 \leqslant x \leqslant \ell,\ x \equiv 0 \ (\mathrm{mod}\ \frac{n}{d})\} \\ &= \#\{y\,;\, 0 \leqslant y \leqslant t-1,\ y \equiv t-\ell-1 \ (\mathrm{mod}\ \frac{n}{d})\}.\end{aligned}$$

The general formula follows from there and the case $t = 1$ is then clear. □

Proposition 2.4. *For any integer $\ell \geqslant 1$ and all $t \in \mathbb{Z}$, we have*

$$\mathrm{core}(M^+_{2\ell}(0,td) \otimes L(v,j)) \cong M^+_{2\ell}(v,j+td) \text{ and}$$
$$\mathrm{core}(M^-_{2\ell}(0,td) \otimes L(v,j)) \cong M^-_{2\ell}(v,j+td).$$

Proof. The proof is by induction on ℓ. Proposition 2.2 shows that the result is true for $\ell = 1$.

Now assume that $\mathrm{core}(M^+_{2\ell}(0,td) \otimes L(v,j)) \cong M^+_{2\ell}(v,j+td)$ for all $t \in \mathbb{Z}$ and for a given $\ell \geqslant 1$. There is a non-split exact sequence

$$0 \to M^+_2(0,td) \to M^+_{2\ell+2}(0,td) \to M^+_{2\ell}(0,(t+1)d) \to 0.$$

Tensoring with $L(v,j)$ gives an exact sequence

$$0 \to M^+_2(v,j+td) \oplus P_1 \to M^+_{2\ell+2}(0,td) \otimes L(v,j) \to M^+_{2\ell}(v,j+(t+1)d) \oplus P_2 \to 0$$

with P_1 and P_2 projective-injective modules, so that, factoring out the split exact sequence $0 \to P_1 \to P_1 \oplus P_2 \to P_2 \to 0$, we have an exact sequence

$$0 \to M_2^+(v, j+td) \to \mathrm{core}(M_{2\ell+2}^+(0,td) \otimes L(v,j)) \oplus P \to M_{2\ell}^+(v, j+(t+1)d) \to 0$$

for some projective module P. Moreover, since $L(v,j)$ is a splitting trace module, by Lemma 2.1 this sequence is not split. By Lemma 2.3, we have $P = 0$ and $\mathrm{core}(M_{2\ell+2}^+(0,td) \otimes L(v,j)) \cong M_{2\ell+2}^+(v, j+td)$ thus proving the induction.

The proof of the isomorphism $\mathrm{core}(M_{2\ell}^-(0,td) \otimes L(v,j)) \cong M_{2\ell}^-(v, j+td)$ is similar, starting with the exact sequence

$$0 \to M_2^-(0,td) \to M_{2\ell+2}^-(0,td) \to M_{2\ell}^-(0,(t-1)d) \to 0. \qquad \square$$

Theorem 2.5. *(cf.* [9, Theorem 4.18]*) Fix an integer $\ell \geqslant 1$. For any (u,i) and (v,j) in $\mathbb{Z}_n^2$ with $2i+u-1 \not\equiv 0 \pmod d$ and $2j+v-1 \not\equiv 0 \pmod d$, we have*

$$\mathrm{core}(M_{2\ell}^+(u,i) \otimes L(v,j)) \cong \bigoplus_{\theta \in \mathfrak{I}} M_{2\ell}^+(u+v, i+j+\theta) \text{ and}$$
$$\mathrm{core}(M_{2\ell}^-(u,i) \otimes L(v,j)) \cong \bigoplus_{\theta \in \mathfrak{I}} M_{2\ell}^-(u+v, i+j+\theta)$$

where $\mathfrak{I}$ is defined in Proposition 1.8.

Proof. Using Propositions 2.4 and 1.8 repeatedly, we have, on the one hand,

$$\begin{aligned} M_{2\ell}^+(0,0) \otimes L(u,i) \otimes L(v,j) &\cong M_{2\ell}^+(0,0) \otimes (L(u,i) \otimes L(v,j)) \\ &\cong M_{2\ell}^+(0,0) \otimes \left(\bigoplus_{\theta \in \mathfrak{I}} L(u+v, i+j+\theta) \oplus P_1 \right) \\ &\cong \bigoplus_{\theta \in \mathfrak{I}} M_{2\ell}^+(0,0) \otimes L(u+v, i+j+\theta) \oplus P_2 \\ &\cong \bigoplus_{\theta \in \mathfrak{I}} M_{2\ell}^+(u+v, i+j+\theta) \oplus P_3 \end{aligned}$$

and on the other hand

$$\begin{aligned} M_{2\ell}^+(0,0) \otimes L(u,i) \otimes L(v,j) &\cong (M_{2\ell}^+(0,0) \otimes L(u,i)) \otimes L(v,j) \\ &\cong (M_{2\ell}^+(u,i) \oplus P_4) \otimes L(v,j) \\ &\cong (M_{2\ell}^+(u,i) \otimes L(v,j)) \oplus P_5 \end{aligned}$$

for some projective modules $P_1, \ldots, P_5$. The projective P_5 is necessarily a direct summand in P_3, therefore the result follows.

The proof of the second isomorphism is similar. $\square$

3. Decomposition of the tensor product of two string modules of even length

In this section, we determine the tensor products of two string modules of even length, in order to provide a correction to the statement and proof of [9, Theorem 4.22].

We proved in [9, Proposition 4.21] that $M_2^+(u,i) \otimes M_2^-(v,j)$ is projective for any (u,i) and (v,j) in $\mathbb{Z}_n^2$. It then follows, using induction on ℓ and t and the Auslander-Reiten sequences for the string modules of even length, that $M_{2\ell}^+(u,i) \otimes M_{2t}^-(v,j)$ is projective for any (u,i) and (v,j) in $\mathbb{Z}_n^2$ and any positive integers ℓ and t.

We denote by $\underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(M,N)$ the space of stable homomorphisms from M to N. We shall need the following isomorphisms.

Lemma 3.1. *Let U, V and W be any $\mathcal{D}(\Lambda_{n,d})$-modules. Then there are isomorphisms*

$$\underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(U \otimes V, W) \cong \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(U, V^* \otimes W)$$
$$\underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(U, V \otimes W) \cong \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(U \otimes V^*, W).$$

Proof. The adjoint functors $- \otimes V$ and $\mathrm{Hom}_k(V,-)$ take projective modules to projective modules, therefore by the Auslander-Kleiner theorem (see [11, 5.3.4]), there is an isomorphism $\mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(U \otimes V, W) \cong \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(U, \mathrm{Hom}_k(V,W))$. Moreover, there is an isomorphism $\mathrm{Hom}_k(V,W) \cong V^* \otimes W$ of $\mathcal{D}(\Lambda_{n,d})$-modules (see [5, Section 4.2.C]), and the first isomorphism follows.

The second isomorphism is a consequence of the first isomorphism and of the isomorphism $V^{**} \cong V$ of $\mathcal{D}(\Lambda_{n,d})$-modules (see [5, Section 4.2.C]). $\square$

We now concentrate on tensor products of the form $M_{2\ell}^+(u,i) \otimes M_{2t}^+(v,j)$ and $M_{2\ell}^-(u,i) \otimes M_{2t}^-(v,j)$. In order to prove our results, we shall need the duals of some modules.

Lemma 3.2. *Assume that $(u,i) \in \mathbb{Z}_n^2$ and $2i+u-1 \not\equiv 0 \pmod d$. Then the following isomorphisms hold.*
(i) $L(u,i)^* \cong L(-u, 1-\sigma_u(i))$.
(ii) $M_{2\ell}^+(u,i)^* \cong M_{2\ell}^+(-u, 1-i-\ell d)$.
(iii) $M_{2\ell}^-(u,i)^* \cong M_{2\ell}^-(-u, 1-i-(\ell-1)d)$.

Proof. We start with the dual of a simple module. It is also a simple module, therefore there exists (v,j) such that $L(u,i)^* \cong L(v,j)$. Set $N = \dim L(u,i) = \dim L(v,j) < d$. We must determine v and j.

The module $L(u,i)$ has a basis $\{\varepsilon_t\,;\, 0 \leqslant t < N\}$ given by $\varepsilon_t = X^t \tilde{F}_{u,i}$, using Proposition 1.2 and the description of $L(u,i)$ preceding it. Let $\{\varepsilon_t^*\,;\, 0 \leqslant t < N\}$ be the dual basis. We have

- $(E_w \cdot \varepsilon_t^*)(\varepsilon_s) = \varepsilon_t^*(S(E_w)\varepsilon_s) = \varepsilon_t^*(E_{-w}\varepsilon_s) = \begin{cases} \varepsilon_t^*(\varepsilon_s) & \text{if } -w = u \\ 0 & \text{otherwise} \end{cases}$
  so that $E_{-u} \cdot \varepsilon_t^* = \varepsilon_t^*$ and $E_w \cdot \varepsilon_t^* = 0$ for $w \neq -u$. By Proposition 1.2 we must have $v = -u$.
- Similarly, we have $X \cdot \varepsilon_t^* = \begin{cases} -q^{-i-t+1}\varepsilon_{t-1}^* & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases}$
  In particular, ε_0^* spans the kernel of the action of X. Then, by Proposition 1.2, the idempotent e_j is the only one which does not annihilate ε_0^*. We can easily check that $e_{1-\sigma_u(i)} \cdot \varepsilon_0^* = \varepsilon_0^*$, therefore $j = 1-\sigma_u(i)$.

We now consider the dual of an indecomposable string module M of even length. Since $M \cong M^{**}$ as $\mathcal{D}(\Lambda_{n,d})$-modules, the dual module M^* is also indecomposable. Moreover, if M has radical length 2, then so has M^* and $\operatorname{top}(M^*) \cong (\operatorname{soc} M)^*$ and $\operatorname{soc}(M^*) \cong (\operatorname{top} M)^*$. We also note that if $x \in \mathbb{Z}_n$ with $x \not\equiv 0 \pmod d$, then $\langle x\rangle + \langle -x\rangle = d$.

First consider $M_2^+(u,i)^*$. We have $\operatorname{soc}(M_2^+(u,i)^*) \cong \operatorname{top}(M_2^+(u,i))^* \cong L(u,i)^* \cong L(-u, 1-\sigma_u(i))$ and $\operatorname{top}(M_2^+(u,i)^*) \cong \operatorname{soc}(M_2^+(u,i))^* \cong L(u,\sigma_u(i))^* \cong L(-u,\sigma_u^2(i)) = L(-u, 1-i-d)$. Note that $\sigma_{-u}(1-i-d) = 1-\sigma_u(i)$. Moreover, $M_2^+(u,i)$ is not projective as a $\Lambda_{n,d}^{*cop}$-module, therefore it has a direct summand of dimension strictly smaller than d. It follows that the $\Lambda_{n,d}^{*cop}$-module $M_2^+(u,i)^*$ also has a direct summand of dimension strictly smaller than d, so that $M_2^+(u,i)^*$ is not projective as a $\Lambda_{n,d}^{*cop}$-module. Therefore $M_2^+(u,i)^* \cong M_2^+(-u, 1-i-d)$.

The dual $M_{2\ell}^+(u,i)^*$ is then obtained by induction, using the Auslander-Reiten sequences and the fact that dualising an Auslander-Reiten sequence gives an Auslander-Reiten sequence.

The proof of (iii) is similar. □

We shall now study the tensor product of string modules of even length, and we start with some special cases, which constitute the initial step in the proof of Theorem 3.4.

Proposition 3.3. *We have*

$$\operatorname{core}(M_{2t}^+(u,i) \otimes M_{2t}^+(v,j)) \cong \bigoplus_{\theta\in\mathfrak{I}} \Big(M_{2t}^+(u+v, i+j+\theta) \oplus M_{2t}^+(u+v, \sigma_{u+v}^{2t-1}(u+v+\theta))\Big)$$
$$\operatorname{core}(M_{2t}^+(u,i) \otimes M_{2(t+1)}^+(v,j)) \cong \bigoplus_{\theta\in\mathfrak{I}} \Big(M_{2t}^+(u+v, i+j+\theta) \oplus M_{2t}^+(u+v, \sigma_{u+v}^{2t+1}(u+v+\theta))\Big)$$
$$\operatorname{core}(M_{2t}^-(u,i) \otimes M_{2t}^-(v,j)) \cong \bigoplus_{\theta\in\mathfrak{I}} \Big(M_{2t}^-(u+v, i+j+\theta) \oplus M_{2t}^-(u+v, \sigma_{u+v}^{-(2t-1)}(u+v+\theta))\Big)$$
$$\operatorname{core}(M_{2t}^-(u,i) \otimes M_{2(t+1)}^-(v,j)) \cong \bigoplus_{\theta\in\mathfrak{I}} \Big(M_{2t}^-(u+v, i+j+\theta) \oplus M_{2t}^-(u+v, \sigma_{u+v}^{-(2t+1)}(u+v+\theta))\Big)$$

where $\mathfrak{I}$ is defined in Proposition 1.8.

Proof. We start by proving the first two isomorphisms in the case $(u,i) = (v,j) = (0,0)$, that is,

$$\begin{cases} \operatorname{core}(M_{2t}^+(0,0) \otimes M_{2t}^+(0,0)) \cong M_{2t}^+(0,0) \oplus M_{2t}^+(0, \sigma_0^{2t-1}(0)) \\ \operatorname{core}(M_{2t}^+(0,0) \otimes M_{2(t+1)}^+(0,0)) \cong M_{2t}^+(0,0) \oplus M_{2t}^+(0, \sigma_0^{2t+1}(0)) \end{cases}$$

Set $\ell = t$ or $\ell = t+1$.

We have an exact sequence

$$0 \to L(0, 1+(\ell-1)d) \to M_{2\ell}^+(0,0) \to \Omega^{\ell-1}(L(0, \sigma_0^{\ell-1}(0))) \to 0.$$

Tensoring on the left with $M_{2t}^+(0,0)$ gives an exact sequence

$$0 \to M_{2t}^+(0, 1+(\ell-1)d) \oplus P_1 \to M_{2t}^+(0,0) \otimes M_{2\ell}^+(0,0) \to \Omega^{\ell-1}(M_{2t}^+(0, \sigma_0^{\ell-1}(0))) \oplus P_2 \to 0$$

for some projective-injective modules P_1 and P_2. It follows that there is an exact sequence

$$0 \to M_{2t}^+(0, 1+(\ell-1)d) \to \operatorname{core}(M_{2t}^+(0,0) \otimes M_{2\ell}^+(0,0)) \oplus P_3 \to M_{2t}^+(0,0) \to 0 \tag{3.1}$$

for some projective module P_3 (obtained by factoring out the split exact sequence $0 \to P_1 \to P_1 \oplus P_2 \to P_2 \to 0$). The exact sequence (3.1) is isomorphic to the following pullback

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{2t}^+(0,1+(\ell-1)d) & \longrightarrow & E & \longrightarrow & M_{2t}^+(0,0) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & M_{2t}^+(0,1+(\ell-1)d) & \longrightarrow & P := \bigoplus_{y=\ell}^{\ell+t-1} P(0,yd) & \xrightarrow{\pi} & \Omega^{-1}(M_{2t}^+(0,1+(\ell-1)d)) \cong M_{2t}^+(0,\ell d) & \longrightarrow & 0
\end{array}$$

where $E = \{(m,p) \in M_{2t}^+(0,0) \times P\,;\, \varphi(m) = \pi(p)\} \cong \mathrm{core}(M_{2t}^+(0,0) \otimes M_{2\ell}^+(0,0)) \oplus P$.

Assume for a contradiction that the sequence (3.1) is not split. Then $\varphi \neq 0$. The modules $M_{2t}^+(0,0)$ and $M_{2t}^+(0,\ell d)$ have the same parity, therefore the image of φ must contain a submodule of $M_{2t}^+(0,\ell d)$ of length 2, that is, $M_2^+(0,\ell d)$. The module $M_2^+(0,\ell d)$ has a generator of the form $\varphi(m) = \pi(e)$ for some $(m,e) \in M_{2t}^+(0,0) \times P(0,\ell d) \subset E$. The submodule of E generated by (m,e) is projective, isomorphic to $P(0,\ell d)$. Since $P(0,\ell d)$ is injective, it is isomorphic to a direct summand of E. Therefore $P(0,\ell d)$, which is a summand in P, is also a summand in $M_{2t}^+(0,0) \otimes M_{2\ell}^+(0,0)$.

Denote by $[M : L]$ the multiplicity of a simple module L as a summand in a semisimple module M. The simple module $L(0,\ell d)$ is in the socle of the projective summand $P(0,\ell d)$ of $M_{2t}^+(0,0) \otimes M_{2\ell}^+(0,0)$ (but does not occur in the socle of $M_{2t}^+(0,0)$), hence it follows from the above that

$$[\mathrm{soc}(\mathrm{core}(M_{2t}^+(0,0) \otimes M_{2\ell}^+(0,0))) : L(0,\ell d)] < [\mathrm{soc}(M_{2t}^+(0,1+(\ell-1)d)) : L(0,\ell d)].$$

We now compute these multiplicities. Set $t = r\frac{n}{d} + s$ with $0 \leqslant s < \frac{n}{d}$. Recall that $\ell \geqslant t$.

We have

$$\begin{aligned}
[\mathrm{soc}(M_{2t}^+(0,1+(\ell-1)d)) : L(0,\ell d)] &= [\bigoplus_{x=\ell}^{\ell+t-1} L(0,xd) : L(0,\ell d)] = [\bigoplus_{x=0}^{t-1} L(0,(x+\ell)d) : L(0,\ell d)] \\
&= \#\{x\,;\, 0 \leqslant x \leqslant t-1,\ x \equiv 0 \ (\mathrm{mod}\ \frac{n}{d})\}.
\end{aligned}$$

Using Lemmas 3.1 and 3.2 and Theorem 2.5, we have

$$\begin{aligned}
[\mathrm{soc}(\mathrm{core}(M_{2t}^+(0,0) \otimes M_{2\ell}^+(0,0))) : L(0,\ell d)] &= \dim \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(L(0,\ell d), \mathrm{core}(M_{2t}^+(0,0) \otimes M_{2\ell}^+(0,0))) \\
&= \dim \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(L(0,\ell d), M_{2t}^+(0,0) \otimes M_{2\ell}^+(0,0)) \\
&= \dim \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(L(0,\ell d) \otimes M_{2\ell}^+(0,0)^*, M_{2t}^+(0,0)) \\
&= \dim \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(L(0,\ell d) \otimes M_{2\ell}^+(0,1-\ell d), M_{2t}^+(0,0)) \\
&= \dim \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(M_{2\ell}^+(0,1), M_{2t}^+(0,0)) \\
&= \dim \mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(M_{2\ell}^+(0,d), M_{2t}^+(0,0))
\end{aligned}$$

which is known by Lemma 2.3.

Finally,

$$\begin{aligned}
\delta &:= [\mathrm{soc}(\mathrm{core}(M_{2t}^+(0,0) \otimes M_{2\ell}^+(0,0))) : L(0,\ell d)] - [\mathrm{soc}(M_{2t}^+(0,1+(\ell-1)d)) : L(0,\ell d)] \\
&= \#\{y\,;\, 0 \leqslant y \leqslant t-1,\ y \equiv 0 \ (\mathrm{mod}\ \frac{n}{d})\} - \#\{y\,;\, 0 \leqslant y \leqslant t-1,\ y \equiv \ell \ (\mathrm{mod}\ \frac{n}{d})\} \\
&\quad + \#\{y\,;\, 0 \leqslant y \leqslant t-1,\ y \equiv t-\ell-1 \ (\mathrm{mod}\ \frac{n}{d})\} - \#\{x\,;\, 0 \leqslant x \leqslant t-1,\ x \equiv 0 \ (\mathrm{mod}\ \frac{n}{d})\} \\
&= \#\{x\,;\, 0 \leqslant x \leqslant t-1,\ x \equiv t-\ell-1 \ (\mathrm{mod}\ \frac{n}{d})\} - \#\{x\,;\, 0 \leqslant x \leqslant t-1,\ x \equiv \ell \ (\mathrm{mod}\ \frac{n}{d})\}.
\end{aligned}$$

We start with the case $\ell = t$. Then

$$\begin{aligned}
0 > \delta &= \#\{x\,;\, 0 \leqslant x \leqslant r\frac{n}{d} + s - 1,\ x \equiv -1 \ (\mathrm{mod}\ \frac{n}{d})\} - \#\{x\,;\, 0 \leqslant x \leqslant r\frac{n}{d} + s - 1,\ x \equiv s \ (\mathrm{mod}\ \frac{n}{d})\} \\
&= r - r = 0
\end{aligned}$$

so that we have a contradiction.

In the case $\ell = t+1$,

$$\begin{aligned}
0 > \delta &= \#\{x\,;\, 0 \leqslant x \leqslant r\frac{n}{d} + s - 1,\ x \equiv -2 \ (\mathrm{mod}\ \frac{n}{d})\} - \#\{x\,;\, 0 \leqslant x \leqslant r\frac{n}{d} + s - 1,\ x \equiv s+1 \ (\mathrm{mod}\ \frac{n}{d})\} \\
&= \begin{cases} (r+1) - (r+1) = 0 & \text{if } s = \frac{n}{d} - 1 \\ r - r = 0 & \text{if } s < \frac{n}{d} - 1 \end{cases}
\end{aligned}$$

so that we have a contradiction.

Therefore the sequence (3.1) is split in both cases and we have

$$\begin{cases} \mathrm{core}(M_{2t}^+(0,0)\otimes M_{2t}^+(0,0)) \cong M_{2t}^+(0,0)\oplus M_{2t}^+(0,\sigma_0^{2t-1}(0)) & (\text{when } \ell = t)\\ \mathrm{core}(M_{2t}^+(0,0)\otimes M_{2(t+1)}^+(0,0)) \cong M_{2t}^+(0,0)\oplus M_{2t}^+(0,\sigma_0^{2t+1}(0)) & (\text{when } \ell = t+1). \end{cases}$$

The result then follows, tensoring these isomorphisms with $L(u,i)$ on the left and $L(v,j)$ on the right, and using the commutativity of the tensor product and Proposition 1.8 on the tensor product of simple modules.

The other two isomorphisms are proved in the same way: there is an exact sequence

$$0 \to M_{2t}^+(0,1-\ell d) \to \mathrm{core}(M_{2t}^-(0,0)\otimes M_{2\ell}^-(0,0))\oplus P_3 \to M_{2t}^-(0,0)\to 0$$

and if the middle term has a projective summand, then it must contain $P(0,-\ell d)$ as a summand. We then compare the multiplicities of the simple module $L(0,-\ell d)$ in $\mathrm{soc}(\mathrm{core}(M_{2t}^-(0,0)\otimes M_{2\ell}^-(0,0)))$ and in $\mathrm{soc}(M_{2t}^-(0,1-\ell d))$. □

We may now rectify [9, Theorem 4.22].

Theorem 3.4. *For any positive integer t and any integer ℓ with $\ell \geqslant t$, we have*

$$\mathrm{core}(M_{2\ell}^+(v,j)\otimes M_{2t}^+(u,i)) \cong \bigoplus_{\theta\in\mathfrak{I}} \left(M_{2t}^+(u+v,i+j+\theta)\oplus M_{2t}^+(u+v,\sigma_{u+v}^{2\ell-1}(i+j+\theta))\right) \text{ and}$$

$$\mathrm{core}(M_{2\ell}^-(v,j)\otimes M_{2t}^-(u,i)) \cong \bigoplus_{\theta\in\mathfrak{I}} \left(M_{2t}^-(u+v,i+j+\theta)\oplus M_{2t}^+(u+v,\sigma_{u+v}^{-(2\ell-1)}(i+j+\theta))\right),$$

where $\mathfrak{I}$ is defined in Proposition 1.8.

Proof. We prove the first isomorphism by induction on ℓ, the proof of the second isomorphism is similar.

By Proposition 3.3, we already know that the result is true for $\ell = t$ and for $\ell = t+1$. Now take $\ell \geqslant t+1$ and assume that the decomposition holds for any ℓ' with $t \leqslant \ell' \leqslant \ell$. We apply Lemma 2.1 with $M = M_{2t}^+(u,i)$ and $C = M_{2\ell}^+(v,j+d)$. The module $M^*\otimes M\otimes C \cong M_{2t}^+(-u,1-i-td)\otimes M_{2t}^+(u,i)\otimes M_{2\ell}^+(v,j+d)$ is the direct sum of a projective module and of indecomposable modules of length $2t<2\ell$ by the induction hypothesis and Proposition 3.3, therefore $M_{2\ell}^+(v,j+d)$ is not a summand in $M_{2t}^+(-u,1-i-td)\otimes M_{2t}^+(u,i)\otimes M_{2\ell}^+(v,j+d)$. It follows that $\mathcal{A}(M_{2\ell}^+(v,j+d))\otimes M_{2t}^+(u,i)$ splits, so that

$$\begin{aligned}\mathrm{core}((M_{2(\ell+1)}^+(v,j)\otimes M_{2t}^+(u,i))\oplus(M_{2(\ell-1)}^+(v,j+d)\otimes M_{2t}^+(u,i)))\\ \cong \mathrm{core}((M_{2\ell}^+(v,j)\otimes M_{2t}^+(u,i))\oplus(M_{2\ell}^+(v,j+d)\otimes M_{2t}^+(u,i))).\end{aligned}$$

Since $t\leqslant \ell-1$, using the induction hypothesis we have

$$\mathrm{core}(M_{2(\ell-1)}^+(v,j+d)\otimes M_{2t}^+(u,i)) \cong \bigoplus_{\theta\in\mathfrak{I}} \left(M_{2t}^+(u+v,i+j+\theta+d)\oplus M_{2t}^+(u+v,\sigma_{u+v}^{2\ell-1}(i+j+\theta))\right)$$

$$\mathrm{core}(M_{2\ell}^+(v,j)\otimes M_{2t}^+(u,i)) \cong \bigoplus_{\theta\in\mathfrak{I}} \left(M_{2t}^+(u+v,i+j+\theta)\oplus M_{2t}^+(u+v,\sigma_{u+v}^{2\ell-1}(i+j+\theta))\right)$$

$$\mathrm{core}(M_{2\ell}^+(v,j+d)\otimes M_{2t}^+(u,i)) \cong \bigoplus_{\theta\in\mathfrak{I}} \left(M_{2t}^+(u+v,i+j+\theta+d)\oplus M_{2t}^+(u+v,\sigma_{u+v}^{2\ell+1}(i+j+\theta))\right)$$

therefore $\mathrm{core}(M_{2(\ell+1)}^+(v,j)\otimes M_{2t}^+(u,i)) \cong \bigoplus_{\theta\in\mathfrak{I}} \left(M_{2t}^+(u+v,i+j+\theta)\oplus M_{2t}^+(u+v,\sigma_{u+v}^{2\ell+1}(i+j+\theta))\right)$ as required.

This concludes the induction. □

4. Decomposition of tensor products involving band modules

In [9], we were unable to give a general expression of the decomposition of the tensor product of a band module with another module. We can do this now, using the methods in the previous two sections.

We consider first the tensor product of a band module of shortest length with a simple module.

Proposition 4.1. *(cf. [9, Proposition 4.17]) For any $\lambda\in k\backslash\{0\}$, there exists $\mu\in k\backslash\{0\}$ such that* $\mathrm{core}(C_\lambda^1(0,0)\otimes L(v,j))\cong C_\mu^1(v,j)$.

Proof. The proof is similar to that of Proposition 2.2. Set $N = \dim L(v,j)$. We tensor the non-split exact sequence $0\to \Omega^{1-\frac{n}{d}}(L(0,\sigma_0^{\frac{n}{d}}(0)))\to C_\lambda^1(0,0)\to L(0,0)\to 0$ by $L(v,j)$. Using Proposition 1.8 as well as $\dim L(0,0)=1$ and $\dim L(0,\sigma_0^{\frac{n}{d}}(0))\in\{1,d-1\}$ depending on the parity of $\frac{n}{d}$ (so that $\#\mathfrak{I}=1$), we obtain an exact sequence $0\to\Omega^{1-\frac{n}{d}}(L(v,\sigma_v^{\frac{n}{d}}(j)))\oplus P\to C_\lambda^1(0,0)\otimes L(v,j)\to L(v,j)\to 0$ with P a projective module or 0. Since P is injective, we have $C_\lambda^1(0,0)\otimes L(v,j)\cong U\oplus P$ for some module U and we obtain an exact sequence $0\to\Omega^{1-\frac{n}{d}}(L(v,\sigma_v^{\frac{n}{d}}(j)))\to U\to L(v,j)\to 0$. The sequence cannot split by Lemma 2.1, so U is an indecomposable module of length $2\frac{n}{d}$, such that $L(v,j)$ is contained in the top of U. Moreover, it follows from Property

1.4 and Remark 1.5 that $C^1_\lambda(0,0)\otimes L(v,j)$, and hence U, is projective as a $\Lambda_{n,d}$-module and as a $\Lambda^{*cop}_{n,d}$-module, therefore we must have $U = C^1_\mu(v,j)$ for some parameter $\mu \in k\backslash\{0\}$. □

As in the case of string modules of even length, this proposition constitutes the initial step in the proof that $\mathrm{core}(C^\ell_\lambda(0,0)\otimes L(v,j))$ is a band module of the form $C^\ell_\mu(v,j)$. In order to continue, we shall need the extensions between band modules.

Lemma 4.2. *Let ℓ and t be positive integers. For any $(u,i)\in\mathbb{Z}^2_n$ with $2i+u-1\not\equiv 0 \pmod d$ and any parameters λ and μ in $k\setminus\{0\}$,*

$$\dim \mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(C^t_\lambda(u,i), C^\ell_\mu(u,i)) = \begin{cases} \min(t,\ell) & \text{if } \lambda=\mu \\ 0 & \text{if } \lambda\neq\mu. \end{cases}$$

Moreover, if $t=1$ and $\lambda=\mu$, the equivalence class of the exact sequence

$$0\to C^\ell_\lambda(u,i)\to C^{\ell+1}_\lambda(u,i)\to C^1_\lambda(u,i)\to 0$$

is a basis of $\mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(C^1_\lambda(u,i), C^\ell_\mu(u,i))$

Proof. There is a parameter $\omega(\lambda)\in k\backslash\{0\}$ such that $\Omega(C^t_\lambda(u,i)) = C^t_{\omega(\lambda)}(u,\sigma_u(i))$. Apply $\mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(-, C^\ell_\mu(u,i))$ to the exact sequence

$$0\to C^t_{\omega(\lambda)}(u,\sigma_u(i))\to \bigoplus_{r=1}^{\frac{n}{d}} P(u,i+rd)^t \to C^t_\lambda(u,i)\to 0$$

to get the exact sequence

$$\begin{aligned} 0\to \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(C^t_\lambda(u,i), C^\ell_\mu(u,i)) &\to \bigoplus_{r=1}^{\frac{n}{d}} \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(P(u,i+rd), C^\ell_\mu(u,i))^t \\ \to \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(C^t_{\omega(\lambda)}(u,\sigma_u(i)), C^\ell_\mu(u,i)) &\to \mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(C^t_\lambda(u,i), C^\ell_\mu(u,i))\to 0. \end{aligned}$$

- Let f be a non-zero map in $\mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(C^t_{\omega(\lambda)}(u,\sigma_u(i)), C^\ell_\mu(u,i))$. Since the modules have different parities, f must map the top $\bigoplus_{x=0}^{\frac{n}{d}-1} L(u,\sigma_u^{2x+1}(i))^t$ of $C^t_{\omega(\lambda)}(u,\sigma_u(i))$ into the socle $\bigoplus_{x=0}^{\frac{n}{d}-1} L(u,\sigma_u^{2x+1}(i))^\ell$ of $C^\ell_\mu(u,i)$, therefore $\dim \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(C^t_{\omega(\lambda)}(u,\sigma_u(i)), C^\ell_\mu(u,i)) = t\ell\frac{n}{d}$.
- $\dim \bigoplus_{r=1}^{\frac{n}{d}} \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(P(u,i+rd), C^\ell_\mu(u,i))^t = t\ell\frac{n}{d}$.
- Therefore $\dim \mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(C^t_\lambda(u,i), C^\ell_\mu(u,i)) = \dim \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(C^t_\lambda(u,i), C^\ell_\mu(u,i))$ and we must prove that

$$\dim \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(C^t_\lambda(u,i), C^\ell_\mu(u,i)) = \begin{cases} 0 & \text{if } \lambda\neq\mu \\ \min(t,\ell) & \text{if } \lambda=\mu. \end{cases}$$

  Both $C^t_\lambda(u,i)$ and $C^\ell_\mu(u,i)$ are modules over the same block $\mathbb{B}_{u,i}$, therefore we may work over the basic algebra $\mathbb{B}_{u,i}$ with band modules such that b_0 is the arrow acting as $J_t(\lambda)$ and $J_\ell(\mu)$ respectively. A morphism $f\in \mathrm{Hom}_{\mathcal{D}(\Lambda_{n,d})}(C^t_\lambda(u,i), C^\ell_\mu(u,i))$ is then completely determined by a linear map $f_0\in \mathrm{Hom}_k(k^t,k^\ell)$ such that $J_\ell(\mu)f_0 = f_0 J_t(\lambda)$. Such a map is zero if $\lambda\neq\mu$, and if $\lambda=\mu$ its matrix is of the form $\begin{pmatrix} A \\ 0_{\ell-t}\end{pmatrix}$ if $t\leqslant\ell$ and $\begin{pmatrix} A & 0_{t-\ell}\end{pmatrix}$ if $t\geqslant\ell$, where A is an upper triangular Toeplitz matrix, that is, of the form $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ 0 & a_1 & \ddots & \vdots \\ \vdots & & \ddots & a_2 \\ 0 & \cdots & \cdots & a_1 \end{pmatrix}$, with $m=\min(\ell,t)$ and $a_1,\dots,a_m$ in k. □

We can now determine the tensor product of a longer band module with a simple module.

Proposition 4.3. *Let ℓ be a positive integer and let λ be in $k\backslash\{0\}$. Let $\mu\in k\backslash\{0\}$ be as in Proposition 4.1. Then*

$$\mathrm{core}(C^\ell_\lambda(0,0)\otimes L(v,j)) \cong C^\ell_\mu(v,j).$$

Proof. The proof is by induction on ℓ. Proposition 4.1 shows that the result is true for $\ell=1$.

Now assume that $\mathrm{core}(C^\ell_\lambda(0,0)\otimes L(v,j))\cong C^\ell_\mu(v,j)$ for a given $\ell\geqslant 1$. There is an exact sequence

$$0\to C^\ell_\lambda(0,0)\to C^{\ell+1}_\lambda(0,0)\to C^1_\lambda(0,0)\to 0.$$

Tensoring with $L(v,j)$ gives an exact sequence

$$0\to C^\ell_\mu(v,j)\oplus P_1\to C^{\ell+1}_\lambda(0,0)\otimes L(v,j)\to C^1_\mu(v,j)\oplus P_2\to 0$$

with P_1 and P_2 projective-injective modules, so that, factoring out the split exact sequence $0 \to P_1 \to P_1 \oplus P_2 \to P_2 \to 0$, we have an exact sequence

$$0 \to C^\ell_\mu(v,j) \to \mathrm{core}(C^{\ell+1}_\lambda(0,0) \otimes L(v,j)) \oplus P \to C^1_\mu(v,j) \to 0$$

for some projective module P. Moreover, since $L(v,j)$ is a splitting trace module, by Lemma 2.1 this sequence is not split. By Lemma 4.2, we have $P = 0$ and $\mathrm{core}(C^{\ell+1}_\lambda(0,0) \otimes L(v,j)) = C^{\ell+1}_\mu(v,j)$ thus proving the induction. □

As we mentioned in Section 1, the parameter λ of the module $C^\ell_\lambda(u,i)$ is not well defined, due to the fact that defining it requires a fixed labelling of the vertices and arrows of the basic algebra of $\mathcal{D}(\Lambda_{n,d})$, which we do not have. Moreover, when computing tensor products, we work with $\mathcal{D}(\Lambda_{n,d})$-modules (and not over the basic algebra). The method in [9, Example 4.23] allows us to determine the parameter μ in specific examples, once the labelling choices are made, but it does not give a general rule.

However, Proposition 4.3 allows us to fix the parameters coherently once and for all in the following way.

Convention on parameters. Define $C^\ell_\lambda(0,0)$ as in Subsection 1.3, with the vertex ϵ_0 of $\mathbb{B}_{0,0}$ corresponding to the trivial module $L(0,0)$. We then set

$$C^\ell_\lambda(v,j) = \mathrm{core}(C^\ell_\lambda(0,0) \otimes L(v,j)).$$

With the definitions recalled in Subsection 1.3 we have $C^\ell_\lambda(u,i) \cong C^\ell_\lambda(u,\sigma^2_u(i)) = C^\ell_\lambda(u,i+d) = \Omega^2(C^\ell_\lambda(u,i))$, and this is compatible with the convention above.

The tensor product of any band module with a simple module is an immediate consequence of this convention.

Proposition 4.4. *Fix an integer $\ell \geqslant 1$ and a parameter $\lambda \in k\backslash\{0\}$. For any (u,i) and (v,j) in $\mathbb{Z}^2_n$ with $2i+u-1 \not\equiv 0 \pmod d$ and $2j+v-1 \not\equiv 0 \pmod d$, we have*

$$\mathrm{core}(C^\ell_\lambda(u,i) \otimes L(v,j)) \cong \bigoplus_{\theta \in \mathfrak{I}} C^\ell_\lambda(u+v, i+j+\theta)$$

where $\mathfrak{I}$ is defined in Proposition 1.8.

***Remark* 4.5.** As a consequence, we know the tensor product of band modules with string modules of odd length up to projectives.

We can now consider the tensor product of band modules with modules of even length. The first result concerns tensor products of band modules with different parameters and tensor products of band modules with string modules of even length.

Proposition 4.6. *Let (u,i) and (v,j) be in $\mathbb{Z}^2_n$ such that d does not divide $2i+u-1$ or $2j+v-1$, let ℓ and t be any positive integers.*

For any λ, μ in $k\backslash\{0\}$ such that $\lambda \neq \mu$, the module $C^\ell_\lambda(u,i) \otimes C^t_\mu(v,j)$ is projective.

For any λ in $k\backslash\{0\}$, the module $C^\ell_\lambda(u,i) \otimes M^\pm_{2t}(v,j)$ is projective.

Proof. We start by proving that $C^1_\lambda(0,0) \otimes C^1_\mu(0,0)$ is projective. First note that for any parameter $\lambda \in k\backslash\{0\}$ there exist parameters $\omega(\lambda), \omega'(\lambda) \in k\backslash\{0\}$ such that $\Omega^{-1}(C^1_\lambda(0,0)) = C^1_{\omega(\lambda)}(0,1)$ and $\Omega^{-1}(C^1_\lambda(0,1)) = C^1_{\omega'(\lambda)}(0,0)$, and moreover that $\omega'(\omega(\lambda)) = \lambda$ so that ω is an injective map.

Assume first that $\frac{n}{d}$ is odd.

- There is an exact sequence $0 \to L(0,\sigma_0^{\frac{n}{d}}(0)) \to \Omega^{-\frac{n}{d}}(L(0,0)) \to C^1_\lambda(0,0) \to 0$. We tensor this with $C^1_\mu(0,0)$ on the right and factor out projective-injectives. Using the fact that $C^1_\mu(0,\sigma_0^{\frac{n}{d}}(0)) \cong C^1_\mu(0,1)$, the resulting exact sequence is

  $$0 \to C^1_\mu(0,1) \xrightarrow[(f_1,f_2)]{} \Omega^{-\frac{n}{d}}(C^1_\mu(0,0)) \oplus \text{ projective} \cong C^1_{\omega(\mu)}(0,1) \oplus P \to C^1_\lambda(0,0) \otimes C^1_\mu(0,0) \to 0.$$

  If $f_1 \neq 0$, then $\omega(\mu) = \mu$ and f_1 is an isomorphism, therefore $C^1_\lambda(0,0) \otimes C^1_\mu(0,0) \cong P$. If $f_1 = 0$, then $C^1_\lambda(0,0) \otimes C^1_\mu(0,0) \cong C^1_{\omega(\mu)}(0,1) \oplus \Omega^{-1}(C^1_\mu(0,1) \oplus$ projective.

  So either $C^1_\lambda(0,0) \otimes C^1_\mu(0,0)$ is projective or $C^1_\lambda(0,0) \otimes C^1_\mu(0,0) \cong C^1_{\omega(\mu)}(0,1) \oplus C^1_{\omega'(\mu)}(0,0) \oplus$ projective.
- Exchanging λ and μ shows that either $C^1_\lambda(0,0) \otimes C^1_\mu(0,0)$ is projective or $C^1_\lambda(0,0) \otimes C^1_\mu(0,0) \cong C^1_{\omega(\lambda)}(0,1) \oplus C^1_{\omega'(\lambda)}(0,0) \oplus$ projective.
- Therefore, since ω is injective, either $\lambda = \mu$ or $C^1_\lambda(0,0) \otimes C^1_\mu(0,0)$ is projective.

If $\frac{n}{d}$ is even, the same arguments work if we start with the exact sequence

$$0 \to C^1_\lambda(0,0) \to \Omega^{\frac{n}{d}}(L(0,0)) \to L(0,\sigma_0^{\frac{n}{d}}(0)) \to 0$$

The proof that $C^1_\lambda(0,0) \otimes M_2^{\pm}(0,xd)$ for $x \in \mathbb{Z}$ is projective is similar, using the exact sequences $0 \to L(0,1+(x-1)d) \to \Omega^{-1}(L(0,xd)) \to M_2^+(0,xd) \to 0$ and $0 \to L(0,1+xd) \to \Omega^{-1}(L(0,xd)) \to M_2^-(0,xd) \to 0$. Then, using induction, the Auslander-Reiten sequences for string and band modules, and the fact that the tensor product of any module by a projective module is projective, it follows that $C^\ell_\lambda(0,0) \otimes C^t_\mu(0,0)$ for $\lambda \neq \mu$ and $C^\ell_\lambda(0,0) \otimes M_{2t}^{\pm}(0,xd)$ are projective.

The general case follows by tensoring with $L(u,i) \otimes L(v,j)$. □

As a consequence, we can give the dual of a band module.

Corollary 4.7. *The dual of $C^\ell_\lambda(u,i)$ is $C^\ell_\lambda(-u,1-i)$.*

Proof. First, as in the case of string modules of even length, and using the fact that the dual of a projective $\Lambda_{n,d}$-module (respectively $\Lambda_{n,d}^{*cop}$-module) is projective because both algebras are self-injective as finite dimensional Hopf algebras, it can be shown that there exists $\mu \in k\backslash\{0\}$ such that $C^\ell_\lambda(u,i)^* \cong C^\ell_\mu(-u,1-i)$. Moreover, we have

$$\begin{aligned} 0 \neq \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(C^\ell_\lambda(u,i), C^\ell_\lambda(u,i)) &\cong \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(C^\ell_\lambda(u,i) \otimes C^\ell_\lambda(u,i)^*, L(0,0)) \\ &\cong \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(C^\ell_\lambda(u,i) \otimes C^\ell_\mu(-u,1-i), L(0,0)) \end{aligned}$$

therefore $C^\ell_\lambda(u,i) \otimes C^\ell_\mu(-u,1-i)$ is not projective and by Proposition 4.6 it follows that $\mu = \lambda$. □

Another consequence is that we can determine the syzygy of any band module.

Corollary 4.8. *Let (u,i) be in $\mathbb{Z}_n^2$ with $2i+u-1 \not\equiv 0 \pmod d$ and let $\lambda \in k\backslash\{0\}$ be a parameter. Then for any positive integer ℓ we have*

$$\Omega(C^\ell_\lambda(u,i)) \cong C^\ell_\lambda(u,\sigma_u(i)) \cong \Omega^{-1}(C^\ell_\lambda(u,i)).$$

Proof. We start with the syzygy of $C^\ell_\lambda(0,0)$. We know that there exists a parameter $\mu \in k\backslash\{0\}$ such that $\Omega(C^\ell_\lambda(0,0)) = C^\ell_\mu(0,1)$. By Lemma 4.2, the space $\mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(C^\ell_\mu(0,1), C^\ell_\mu(0,1))$ is not zero. Moreover, there are isomorphisms

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(C^\ell_\mu(0,1), C^\ell_\mu(0,1)) &\cong \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(\Omega(C^\ell_\mu(0,1)), C^\ell_\mu(0,1) \otimes L(0,0)) \\ &\cong \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(\Omega^2(C^\ell_\lambda(0,0)) \otimes C^\ell_\mu(0,1)^*, L(0,0)) \\ &\cong \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(C^\ell_\lambda(0,0) \otimes C^\ell_\mu(0,0), L(0,0)) \end{aligned}$$

and therefore $C^\ell_\lambda(0,0) \otimes C^\ell_\mu(0,0)$ is not projective. It follows from Proposition 4.6 that $\mu = \lambda$ and therefore that $\Omega(C^\ell_\lambda(0,0)) = C^\ell_\lambda(0,1)$.

We then have the following isomorphisms.

$$\begin{aligned} \Omega(C^\ell_\lambda(u,i)) &\cong \Omega(\mathrm{core}(C^\ell_\lambda(0,0) \otimes L(u,i))) \cong \mathrm{core}(\Omega(C^\ell_\lambda(0,0) \otimes L(u,i))) \\ &\cong \mathrm{core}(\Omega(C^\ell_\lambda(0,0)) \otimes L(u,i)) \cong \mathrm{core}(C^\ell_\lambda(0,1) \otimes L(u,i)) \\ &\cong \mathrm{core}(C^\ell_\lambda(0,0) \otimes L(0,1) \otimes L(u,i)) \cong \mathrm{core}(C^\ell_\lambda(0,0) \otimes L(u,\sigma_u(i))) \cong C^\ell_\lambda(u,\sigma_u(i)). \end{aligned}$$

Finally, since $\Omega^2(C^\ell_\lambda(u,i)) \cong C^\ell_\lambda(u,i)$, we also have $\Omega^{-1}(C^\ell_\lambda(u,i)) \cong \Omega(C^\ell_\lambda(u,i))$. □

The next result gives the tensor product of any two band modules with the same parameter up to projectives.

Theorem 4.9. *For any positive integer t and any integer ℓ with $\ell \geqslant t$, we have*

$$\mathrm{core}(C^\ell_\lambda(v,j) \otimes C^t_\lambda(u,i)) \cong \bigoplus_{\theta \in \mathfrak{I}} \left(C^t_\lambda(u+v, i+j+\theta) \oplus C^t_\lambda(u+v, \sigma_{u+v}(i+j+\theta)) \right).$$

where $\mathfrak{I}$ is defined in Proposition 1.8.

Proof. We prove the isomorphism in the case $(u,i) = (v,j) = (0,0)$, that is,

$$\mathrm{core}(C^\ell_\lambda(0,0) \otimes C^t_\lambda(0,0)) \cong C^t_\lambda(0,0) \oplus C^t_\lambda(0,1)$$

and the result follows by taking the tensor product with $L(u,i) \otimes L(v,j)$.

By working with modules over the basic algebra, we can see that there is an exact sequence

$$0 \to L(0,1-d) \to C^\ell_\lambda(0,0) \to \Omega^{\ell\frac{n}{d}-1}(L(0,\sigma_0^{\ell\frac{n}{d}-1}(0))) \to 0.$$

Tensoring on the right with $C_\lambda^t(0,0)$ gives an exact sequence

$$0 \to C_\lambda^t(0,1) \oplus P_1 \to C_\lambda^\ell(0,0) \otimes C_\lambda^t(0,0) \to \Omega^{\ell\frac{n}{d}-1}(C_\lambda^t(0,\sigma_0^{\ell\frac{n}{d}-1}(0))) \oplus P_2 \to 0$$

for some projective-injective modules P_1 and P_2. It follows that there is an exact sequence

$$0 \to C_\lambda^t(0,1) \to \mathrm{core}(C_\lambda^\ell(0,0) \otimes C_\lambda^t(0,0)) \oplus P_3 \to C_\lambda^t(0,0) \to 0 \tag{4.1}$$

for some projective module P_3 (obtained by factoring out the split exact sequence $0 \to P_1 \to P_1 \oplus P_2 \to P_2 \to 0$). The exact sequence (4.1) is isomorphic to the following pullback

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_\lambda^t(0,1) & \longrightarrow & E & \longrightarrow & C_\lambda^t(0,0) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & C_\lambda^t(0,1) & \longrightarrow & P := \bigoplus_{y=0}^{\frac{n}{d}-1} P(0,yd)^t & \xrightarrow{\pi} & \Omega^{-1}(C_\lambda^t(0,1)) \cong C_\lambda^t(0,0) & \longrightarrow & 0
\end{array}$$

where $E = \{(m,p) \in C_\lambda^t(0,0) \times P\,;\, \varphi(m) = \pi(p)\} \cong \mathrm{core}(C_\lambda^\ell(0,0) \otimes C_\lambda^t(0,0)) \oplus P$.

Assume for a contradiction that the sequence (4.1) is not split. Then $\varphi \neq 0$ so that φ is a non-zero endomorphism of $C_\lambda^t(0,0)$. Let $m \in C_\lambda^t(0,0)$ be an element that is not in the radical and such that $\varphi(m) \neq 0$, we may assume that the submodule generated by m has a simple top. Then $\varphi(m) = \pi(e)$ where $e \in P$ and e generates an indecomposable projective summand. The element (m,e) belongs to E and it generates an indecomposable projective module. It is then a summand of E since projectives are injective. Let this summand be $P(0,yd)$ for some y, it is then also a summand in $C_\lambda^\ell(0,0) \otimes C_\lambda^t(0,0)$. Denote by $[M : L]$ the multiplicity of a simple module L as a summand in a semisimple module M. The simple module $L(0,yd)$ is in the socle of the projective summand $P(0,yd)$ of $C_\lambda^\ell(0,0) \otimes C_\lambda^t(0,0)$ (but does not occur in the socle of $C_\lambda^t(0,0)$), hence it follows from the above that

$$[\mathrm{soc}(\mathrm{core}(C_\lambda^\ell(0,0) \otimes C_\lambda^t(0,0))) : L(0,yd)] < [\mathrm{soc}\, C_\lambda^t(0,1) : L(0,yd)] = t.$$

Using Lemma 3.1, Proposition 4.3, Corollary 4.7 and the fact that $C_\lambda^t(0,1+yd) \cong C_\lambda^t(0,1)$, we have

$$\begin{aligned}
[\mathrm{soc}(\mathrm{core}(C_\lambda^\ell(0,0) \otimes C_\lambda^t(0,0))) : L(0,yd)] &= \dim \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(L(0,yd), \mathrm{core}(C_\lambda^\ell(0,0) \otimes C_\lambda^t(0,0))) \\
&= \dim \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(L(0,yd), C_\lambda^\ell(0,0) \otimes C_\lambda^t(0,0)) \\
&= \dim \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(L(0,yd) \otimes C_\lambda^t(0,0)^*, C_\lambda^\ell(0,0)) \\
&= \dim \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(C_\lambda^t(0,1), C_\lambda^\ell(0,0)) \\
&= \dim \mathrm{Ext}^1_{\mathcal{D}(\Lambda_{n,d})}(C_\lambda^t(0,0), C_\lambda^\ell(0,0)) = t
\end{aligned}$$

by Lemma 4.2 since $t \leqslant \ell$. Therefore we have obtained a contradiction, the sequence (4.1) splits, and the result follows. □

In many of our proofs, we have worked with $\mathbb{B}_{0,0}$-modules, then tensored with non-projective simple modules to obtain the general result we were seeking. This approach can be formalised as follows.

Theorem 4.10. *Let $L(u,i)$ be a non-projective simple module. Then $- \otimes L(u,i)$ induces a stable equivalence between $\mathbb{B}_{0,0}$ and $\mathbb{B}_{u,i}$.*

The proof uses the following lemma.

Lemma 4.11. *Let $L(u,i)$ be a non-projective simple module of dimension N. There is an isomorphism*

$$\mathrm{core}(L(u,i) \otimes L(u,i)^*) \cong \bigoplus_{\tau=1-\min(N,d-N)}^{0} L(0,\tau),$$

and the blocks $\mathbb{B}_{0,\tau}$, with $1 - \min(N, d-N) \leqslant \tau \leqslant 0$, are pairwise distinct.

Proof. The isomorphism follows from Lemma 3.2 which states that $L(u,i)^* \cong L(-u, 1-\sigma_u(i))$, Proposition 1.8, and the fact that $1 - \sigma_u(i) + i = 1 - N$. Since $1 - \min(N, d-N) > -\frac{d}{2}$, we need only prove that the blocks $\mathbb{B}_{0,\tau}$ with $-\frac{d}{2} < \tau \leqslant 0$ are pairwise different.

The block $\mathbb{B}_{0,\tau}$ contains precisely the simple modules $L(0,j)$ with j in the σ_0-orbit of τ, that is, $j \in \{\tau + td, \sigma_0(\tau) + td\,;\, 0 \leqslant t < \frac{n}{d}\}$ (recall that j is taken modulo n). There are precisely two representatives of the σ_0-orbit of τ in $]-d, 0]$. Moreover, if $-d < j \leqslant -\frac{d}{2}$, then $-1-2d < 2j-1 \leqslant -d-1$ so $\sigma_0(j) = d + j - \langle 2j-1 \rangle = d + j - (2j-1+2d) = -j-d+1$, and $-\frac{d}{2}+1 \leqslant -j-d+1 < 1$ therefore $-\frac{d}{2} < \sigma_0(j) \leqslant 0$. Similarly, if $-\frac{d}{2} < j \leqslant 0$ then $-d < \sigma_0^{-1}(j) = \sigma_0(j) - d \leqslant -\frac{d}{2}$. It follows that the σ_0-orbit of τ has precisely one representative in $]-\frac{d}{2}, 0]$, which proves our claim. □

Proof of Theorem 4.10. Define a functor $F : \mathbb{B}_{0,0}\text{-}\underline{\mathrm{mod}} \to \mathcal{D}(\Lambda_{n,d})\text{-}\underline{\mathrm{mod}}$ by $F(M) = \mathrm{core}(M \otimes L(u,i))$ on objects and $F(f) = f \otimes \mathrm{id}_{L(u,i)}$ restricted and co-restricted to the cores of the modules on morphisms.

We first determine the image of a simple module under F. The simple modules in $\mathbb{B}_{0,0}$ are the modules $L(0,td)$, which have dimension 1, and the modules $L(0,1+td)$, which have dimension $d-1$, for $0 \leqslant t < \frac{n}{d}$. By Proposition 1.8, we have

$$F(L(0,td)) = \mathrm{core}(L(0,td) \otimes L(u,i)) \cong L(u,i+td) = L(u,\sigma_u^{2t}(i))$$
$$F(L(0,1+td)) = \mathrm{core}(L(0,1+td) \otimes L(u,i)) \cong L(u,\sigma_u(i)+td) = L(u,\sigma_u^{2t+1}(i))$$

which are simple $\mathbb{B}_{u,i}$-modules. By induction on the length of modules, it follows that F sends any non-projective $\mathbb{B}_{0,0}$-module to a non-projective $\mathbb{B}_{u,i}$-module. Therefore F induces a functor $\mathbb{B}_{0,0}\text{-}\underline{\mathrm{mod}} \to \mathbb{B}_{u,i}\text{-}\underline{\mathrm{mod}}$ which we denote also by F.

We can be more specific. We have the following:

$$F(M_{2\ell}^{\pm}(0,0)) = \mathrm{core}(M_{2\ell}^{\pm}(0,0) \otimes L(u,i)) \cong M_{2\ell}^{\pm}(u,i)$$
$$F(C_{\lambda}^{\ell}(0,0)) = \mathrm{core}(C_{\lambda}^{\ell}(0,0) \otimes L(u,i)) \cong C_{\lambda}^{\ell}(u,i).$$

Now note that the functor $- \otimes L(u,i)$ defined on $\mathcal{D}(\Lambda_{n,d})$-mod takes projectives to projectives and is exact, therefore it commutes with Ω. It follows that F commutes with Ω. Therefore we can give the image by F of any non-projective indecomposable module:

$$F(\Omega^m(L(0,\sigma_0^t(0)))) \cong \Omega^m(F(L(0,\sigma_0^t(0)))) \cong \Omega^m(L(u,\sigma_u^t(i))),$$
$$F(M_{2\ell}^{\pm}(0,\sigma_0^t(0))) \cong F(\Omega^{-(\pm t)}(M_{2\ell}^{\pm}(0,0))) \cong \Omega^{-(\pm t)}(F(M_{2\ell}^{\pm}(0,0))) \cong \Omega^{-(\pm t)}(M_{2\ell}^{\pm}(u,i)) \cong M_{2\ell}^{\pm}(u,\sigma_u^t(i)),$$
$$F(C_{\lambda}^{\ell}(0,\sigma_0^t(0))) \cong F(\Omega^{-t}(C_{\lambda}^{\ell}(0,0))) \cong \Omega^{-t}(F(C_{\lambda}^{\ell}(0,0))) \cong \Omega^{-t}(C_{\lambda}^{\ell}(u,i)) \cong C_{\lambda}^{\ell}(u,\sigma_u^t(i)).$$

Consequently, for any $M' \in \mathbb{B}_{u,i}\text{-}\underline{\mathrm{mod}}$, there exists $M \in \mathbb{B}_{0,0}\text{-}\underline{\mathrm{mod}}$ such that $M' \cong F(M)$.

Finally, in order to prove that F is indeed an equivalence of categories, we must prove that for any modules M_1 and M_2 in $\mathbb{B}_{0,0}$-$\underline{\mathrm{mod}}$, we have $\underline{\mathrm{Hom}}_{\mathbb{B}_{u,i}}(F(M_1),F(M_2)) \cong \underline{\mathrm{Hom}}_{\mathbb{B}_{0,0}}(M_1,M_2)$. We have

$$\begin{aligned}
\underline{\mathrm{Hom}}_{\mathbb{B}_{u,i}}(F(M_1),F(M_2)) &\cong \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(M_1 \otimes L(u,i), M_2 \otimes L(u,i)) \\
&\cong \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(M_1 \otimes L(u,i) \otimes L(u,i)^*, M_2) \text{ by Lemma 3.1} \\
&\cong \bigoplus_{\tau=1-\min(N,1-N)}^{0} \underline{\mathrm{Hom}}_{\mathcal{D}(\Lambda_{n,d})}(M_1 \otimes L(0,\tau), M_2) \text{ by Lemma 4.11} \\
&\cong \bigoplus_{\tau=1-\min(N,1-N)}^{0} \underline{\mathrm{Hom}}_{\mathbb{B}_{0,0}}(M_1 \otimes L(0,\tau), M_2)
\end{aligned}$$

since M_2 belongs to $\mathbb{B}_{0,0}$. However, by Lemma 4.11 and the beginning of this proof, the modules $M_1 \otimes L(0,\tau)$ belong to pairwise distinct blocks, and the only one belonging to $\mathbb{B}_{0,0}$ is $M_1 \otimes L(0,0) \cong M_1$. Therefore $\underline{\mathrm{Hom}}_{\mathbb{B}_{u,i}}(F(M_1),F(M_2)) \cong \underline{\mathrm{Hom}}_{\mathbb{B}_{0,0}}(M_1,M_2)$. □

5. Description of the stable Green ring of $\mathcal{D}(\Lambda_{n,d})$

Combining the results in this paper with the results in [9], we now have a complete description of the stable Green ring of $\mathcal{D}(\Lambda_{n,d})$, which we give in Table 1 on page 17.

The stable Green ring is commutative so that we may assume for instance that $\ell \geqslant t$ in Table 1.

6. Application to endotrivial modules and algebraic modules

Endotrivial modules have been classified and used in the context of group algebras, see [2]. Here, we determine all the endotrivial modules over $\mathcal{D}(\Lambda_{n,d})$.

Definition 6.1. *An endotrivial module over $\mathcal{D}(\Lambda_{n,d})$ is a $\mathcal{D}(\Lambda_{n,d})$-module M such that* $\mathrm{core}(M \otimes M^*) \cong L(0,0)$ *(the trivial module).*

***Remark* 6.2.** An endotrivial module is necessarily a splitting trace module. It follows from the beginning of Section 2 that any indecomposable endotrivial $\mathcal{D}(\Lambda_{n,d})$-module must have odd length.

Proposition 6.3. *The indecomposable endotrivial modules over $\mathcal{D}(\Lambda_{n,d})$ are the simple modules of dimension* 1 *and* $d-1$ *and their syzygies.*

Proof. As mentioned in the remark above, if M is an endotrivial module, then there exist a non-projective simple module L and an integer $m \in \mathbb{Z}$ such that $M \cong \Omega^m(L)$. Moreover,

$$\mathrm{core}(M \otimes M^*) \cong \mathrm{core}(\Omega^m(L) \otimes \Omega^{-m}(L^*)) \cong \mathrm{core}(\Omega^{m-m}(L \otimes L^*)) \cong \mathrm{core}(L \otimes L^*)$$

so that M is endotrivial if, and only if, L is endotrivial.

$\otimes$	$\Omega^m(L(v,j))$	$M_{2\ell}^+(v,j)$	$M_{2\ell}^-(v,j)$	$C_\mu^\ell(v,j)$
$\Omega^n(L(u,i))$	$\bigoplus_{\theta\in\mathfrak{I}} \Omega^{m+n}(L(w,i+j+\theta))$ [9, Theorem 4.1 and § 4.2]	$\bigoplus_{\theta\in\mathfrak{I}} M_{2\ell}^+(w,\sigma_w^{-n}(i+j+\theta))$ Theorem 2.5	$\bigoplus_{\theta\in\mathfrak{I}} M_{2\ell}^-(w,\sigma_w^{n}(i+j+\theta))$ Theorem 2.5	if n is even: $\bigoplus_{\theta\in\mathfrak{I}} C_\mu^\ell(w,i+j+\theta)$ if n is odd: $\bigoplus_{\theta\in\mathfrak{I}} C_\mu^\ell(w,\sigma_w(i+j+\theta))$ Proposition 4.4
$M_{2t}^+(u,i)$	$\bigoplus_{\theta\in\mathfrak{I}} M_{2t}^+(w,\sigma_w^{-m}(i+j+\theta))$ Theorem 2.5	$\bigoplus_{\theta\in\mathfrak{I}} \Big(M_{2t}^+(w,i+j+\theta)\oplus M_{2t}^+(w,\sigma_w^{2\ell-1}(i+j+\theta))\Big)$ Theorem 3.4	0 [9, Proposition 4.21]	0 Proposition 4.6
$M_{2t}^-(u,i)$	$\bigoplus_{\theta\in\mathfrak{I}} M_{2t}^-(w,\sigma_w^{m}(i+j+\theta))$ Theorem 2.5	0 [9, Proposition 4.21]	$\bigoplus_{\theta\in\mathfrak{I}} \Big(M_{2t}^-(w,i+j+\theta)\oplus M_{2t}^-(w,\sigma_w^{-(2\ell-1)}(i+j+\theta))\Big)$ Theorem 3.4	0 Proposition 4.6
$C_\lambda^t(u,i)$	if m is even: $\bigoplus_{\theta\in\mathfrak{I}} C_\lambda^t(w,i+j+\theta)$ if m is odd: $\bigoplus_{\theta\in\mathfrak{I}} C_\lambda^t(w,\sigma_w(i+j+\theta))$ Proposition 4.4	0 Proposition 4.6	0 Proposition 4.6	if $\lambda=\mu$: $\bigoplus_{\theta\in\mathfrak{I}} \Big(C_\lambda^t(w,i+j+\theta)\oplus C_\lambda^t(w,\sigma_w(i+j+\theta))\Big)$ otherwise: 0 Theorem 4.9

TABLE 1. Description of the product in the stable Green ring of $\mathcal{D}(\Lambda_{n,d})$

where, for (u,i) and (v,j) in $\mathbb{Z}_n^2$, we have put

$$\mathfrak{I}=\begin{cases}\{\theta\,;\,0\leqslant\theta\leqslant\min(\dim L(u,i),\dim L(v,j))-1\} \text{ if } \dim L(u,i)+\dim L(v,j)\leqslant d\\ \{\theta\,;\,\dim L(u,i)+\dim L(v,j)-d\leqslant\theta\leqslant\min(\dim L(u,i),\dim L(v,j))-1\} \text{ otherwise.}\end{cases}$$

and $w=u+v$. We assume that $\ell\geqslant t$.

Set $L = L(u,i)$ and $N = \dim L(u,i)$. By Lemma 4.11,

$$\operatorname{core}(L \otimes L^*) \cong \bigoplus_{\tau=1-\min(N,d-N)}^{0} L(0,\tau),$$

therefore L is endotrivial if and only if $\min(N, d-N) = 1$, that is, $N = 1$ or $N = d-1$. □

***Remark* 6.4.** A simple module belonging to the block $\mathbb{B}$ is endotrivial if, and only if, all the modules of odd length that belong to $\mathbb{B}$ are endotrivial.

We now classify algebraic modules.

Definition 6.5. *An indecomposable module M is algebraic if there are only finitely many non-isomorphic indecomposable summands in $T(M) = \bigoplus_{t\geqslant 1} M^{\otimes t}$.*

***Remark* 6.6.** The tensor product of projective modules is again projective and there are only finitely many isomorphism classes of projective $\mathcal{D}(\Lambda_{n,d})$-modules, therefore any projective $\mathcal{D}(\Lambda_{n,d})$-module is algebraic.

Proposition 6.7. *A non-projective indecomposable $\mathcal{D}(\Lambda_{n,d})$-module is algebraic if and only if it is simple or of even length.*

Proof. Let M be a non-projective indecomposable $\mathcal{D}(\Lambda_{n,d})$-module. Since there are only finitely many isomorphism classes of projective $\mathcal{D}(\Lambda_{n,d})$-modules, we can ignore the projective summands that appear when taking repeated tensor products of M.

- If $M = L$ is a simple module, all the non-projective indecomposable summands in $L^{\otimes t}$ are simple. Since there are only finitely many non-isomorphic simple modules, L is algebraic.
- If $M = \Omega^{\ell}(L)$ for some simple module L and $\ell \in \mathbb{Z}$ non-zero, then M is not algebraic. Indeed, we have $\operatorname{core}(M^{\otimes t}) \cong \Omega^{\ell t}(\operatorname{core}(L^{\otimes t}))$ and when t varies, we get infinitely many non-isomorphic indecomposable summands.
- If $M = M^{+}_{2\ell}(u,i)$, all the non-projective indecomposable summands in $M^{\otimes t}$ are of the form $M^{+}_{2\ell}(v,j)$ with the same ℓ. There are only finitely many such modules up to isomorphism, therefore $M^{+}_{2\ell}(u,i)$ is algebraic. Similarly, $M^{-}_{2\ell}(u,i)$ is algebraic.
- If $M = C^{\ell}_{\lambda}(u,i)$, all the non-projective indecomposable summands in $M^{\otimes t}$ are of the form $C^{\ell}_{\lambda}(v,j)$ with the same ℓ and λ. There are only finitely many such modules up to isomorphism, therefore $C^{\ell}_{\lambda}(u,i)$ is algebraic. □

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K. ERDMANN, MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, ANDREW WILES BUILDING, ROQ, WOODSTOCK ROAD, OXFORD OX2 6GG, UNITED KINGDOM
E-mail address: erdmann@maths.ox.ac.uk

E.L. GREEN, MATHEMATICS DEPARTMENT, VIRGINIA TECH, BLACKSBURG, VA 24061-0123, USA
E-mail address: resolme@math.vt.edu

N. SNASHALL, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LEICESTER, UNIVERSITY ROAD, LEICESTER LE1 7RH, UNITED KINGDOM
E-mail address: njs5@le.ac.uk

R. TAILLEFER, UNIVERSITÉ CLERMONT AUVERGNE, CNRS, LMBP, F-63000 CLERMONT-FERRAND, FRANCE
E-mail address: Rachel.Taillefer@uca.fr