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▶ **To cite this version:**

Pramod N. Achar, Anthony Henderson, Daniel Juteau, Simon Riche. Weyl group actions on the Springer sheaf. Proceedings of the London Mathematical Society, 2014, 108, pp.1501-1528. hal-00813156

HAL Id: hal-00813156
https://hal.science/hal-00813156

Submitted on 15 Apr 2013

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WEYL GROUP ACTIONS ON THE SPRINGER SHEAF

PRAMOD N. ACHAR, ANTHONY HENDERSON, DANIEL JUTEAU, AND SIMON RICHE

Abstract. We show that two Weyl group actions on the Springer sheaf with arbitrary coefficients, one defined by Fourier transform and one by restriction, agree up to a twist by the sign character. This generalizes a familiar result from the setting of ℓ-adic cohomology, making it applicable to modular representation theory. We use the Weyl group actions to define a Springer correspondence in this generality, and identify the zero weight spaces of small representations in terms of this Springer correspondence.

1. Introduction

1.1. In the 1970s, T.A. Springer made the surprising discovery [Spr] that the Weyl group $\mathbf{W}$ of a reductive group G acts in a natural way on the ℓ-adic cohomology of Springer fibres. Shortly thereafter, Lusztig [L1] gave another construction of such an action, using the brand-new language of perverse sheaves. It is a fact of fundamental importance that these two constructions almost coincide: they differ only by the sign character of $\mathbf{W}$, as shown in [Ho]. Thus there is essentially only one Springer correspondence over a field of characteristic zero. The aim of this paper is to prove a similar result with arbitrary coefficients.

The study of Springer theory with coefficients in a local ring or a finite field was initiated by the third author in his thesis [Ju1]. That work, following [HK, Bry], constructed the $\mathbf{W}$-action via a kind of Fourier transform (as did Springer, although his construction was different). On the other hand, in the later paper [AHR], the other authors worked with a $\mathbf{W}$-action defined as in [L1, BM]. One purpose of this paper is to fill in the relationship between [Ju1] and [AHR], and hence derive further consequences of the results of [AHR]. The main result of this paper is also used by Mautner in his geometric study of Schur–Weyl duality [Ma].

1.2. A precise statement is as follows. Let $\Bbbk$ be a commutative noetherian ring of finite global dimension. The *Springer sheaf with coefficients in* $\Bbbk$, denoted $\underline{\mathsf{Spr}}(\Bbbk)$, is the perverse sheaf on the nilpotent cone $\mathcal{N}$ of G obtained by pushing forward the appropriate shift of the constant sheaf $\underline{\Bbbk}$ along the Springer resolution. (For full definitions, see Section 2.) There are two actions of $\mathbf{W}$ on $\underline{\mathsf{Spr}}(\Bbbk)$, giving rise to

$$\dot{\varphi} : \Bbbk[\mathbf{W}] \to \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk)) \qquad \text{given by 'Fourier transform' as in [Bry, Ju1],}$$
$$\dot{\rho} : \Bbbk[\mathbf{W}] \to \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk)) \qquad \text{given by 'restriction' as in [BM, AHR].}$$

2010 *Mathematics Subject Classification.* Primary 17B08, 20G05; Secondary 14M15.

P.A. was supported by NSF Grant No. DMS-1001594. A.H. was supported by ARC Future Fellowship Grant No. FT110100504. D.J. was supported by ANR Grant No. ANR-09-JCJC-0102-01. S.R. was supported by ANR Grants No. ANR-09-JCJC-0102-01 and ANR-2010-BLAN-110-02.

It follows from general properties of the Fourier transform that $\dot{\varphi}$ is a ring isomorphism. Next, let $\varepsilon : \mathbf{W} \to \{\pm 1\}$ be the sign character, and let

$$\mathsf{sgn} : \Bbbk[\mathbf{W}] \to \Bbbk[\mathbf{W}] \qquad \text{be given by} \qquad \mathsf{sgn}(w) = \varepsilon(w)w \qquad \text{for } w \in \mathbf{W}.$$

The main result of the paper is the following.

Theorem 1.1. *We have* $\dot{\rho} = \dot{\varphi} \circ \mathsf{sgn}$.

An immediate corollary is that $\dot{\rho}$ is also an isomorphism, generalizing the result for $\Bbbk = \mathbb{Q}$ proved by Borho and MacPherson [BM, Théorème 3]. (An alternative proof of this corollary will be given by the fourth author in [Ri].) Further consequences, concerning the Springer correspondence induced by $\dot{\varphi}$ (or equivalently by $\dot{\rho}$) and its relation to the zero weight spaces of small representations, are obtained in Section 5.

1.3. The 'easy' case of Theorem 1.1 is when the following condition holds: $\Bbbk$ embeds either in a $\mathbb{Q}$-algebra, or in an $\mathbb{F}_p$-algebra where p does not divide $|\mathbf{W}|$. Under this assumption, Theorem 1.1 can be proved by arguments that have already appeared in the ℓ-adic setting; we explain this proof in Section 4. The main idea is a comparison with the classical $\mathbf{W}$-action on the cohomology of the flag variety $\mathscr{B}$ of G. The condition on $\Bbbk$ is needed to ensure that $H^\bullet(\mathscr{B}; \Bbbk)$ is a faithful $\Bbbk[\mathbf{W}]$-module. Note that $\Bbbk = \mathbb{Z}$ satisfies the condition, but the case of most interest in modular representation theory, when $\Bbbk$ is a field of characteristic dividing $|\mathbf{W}|$, does not.

We will deduce the general case from the case $\Bbbk = \mathbb{Z}$. In order to do so, one might try to use the morphism $\mathrm{End}(\underline{\mathsf{Spr}}(\mathbb{Z})) \to \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk))$ induced by the 'extension of scalars' functor $(-) \overset{L}{\otimes}_{\mathbb{Z}} \Bbbk : D^{\mathrm{b}}(\mathcal{N}, \mathbb{Z}) \to D^{\mathrm{b}}(\mathcal{N}, \Bbbk)$. However, to follow this strategy one would have to check that many isomorphisms between sheaf functors are compatible with extension of scalars. It turns out to be more convenient to use 'restriction of scalars' (or forgetful) functors $\mathrm{For}_{\mathbb{Z},\Bbbk}$. One slight drawback is that there is no obvious morphism $\mathrm{End}(\underline{\mathsf{Spr}}(\mathbb{Z})) \to \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk))$ induced by $\mathrm{For}^{\mathcal{N}}_{\mathbb{Z},\Bbbk}$. To bypass this difficulty we regard $\underline{\mathsf{Spr}}^{\Bbbk}$ as a *functor* from the category of $\Bbbk$-modules to the category of perverse sheaves on $\mathcal{N}$, and we study the ring $\mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk})$ of natural endotransformations of this functor. The main ideas of Springer theory can be developed in this functorial setting. A functor is more 'rigid' than a single perverse sheaf, which allows us to define the desired morphism $\mathrm{End}(\underline{\mathsf{Spr}}^{\mathbb{Z}}) \to \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk})$ compatible with the functorial versions of $\dot{\varphi}$ and $\dot{\rho}$. That compatibility enables us to deduce Theorem 1.1 from the $\Bbbk = \mathbb{Z}$ case.

1.4. Outline of the paper. Section 2 contains the basic constructions of Springer theory in our functorial setting. Section 3 contains the argument that reduces Theorem 1.1 to the $\Bbbk = \mathbb{Z}$ case. In Section 4, we explain how to prove Theorem 1.1 in the 'easy' case, thus completing the proof. Section 5 presents the applications to the Springer correspondence and to small representations. Finally, Section 6 discusses an analogue of Theorem 1.1 for the étale topology over a finite field. Appendix A collects some general lemmas expressing the compatibility of various isomorphisms of sheaf functors.

1.5. Acknowledgements. We are grateful to Frank Lübeck for supplying data about small representations in positive characteristic that supplemented [Lü].

2. Preliminaries

2.1. Groups and varieties. Let G be a connected reductive algebraic group over $\mathbb{C}$, with Lie algebra $\mathfrak{g}$. Let $\mathscr{B}$ be the flag variety, which parametrizes Borel subgroups of G. If $B \in \mathscr{B}$, we let U_B be its unipotent radical. Let $\mathfrak{H}$ be the universal Cartan subalgebra of $\mathfrak{g}$. Recall that if B is a Borel subgroup of G, $\mathfrak{H}$ is canonically isomorphic to $\mathrm{Lie}(B)/\mathrm{Lie}(U_B)$. Let $\mathbf{W}$ be the universal Weyl group; by definition, it acts naturally on $\mathfrak{H}$.

Consider the vector bundle

$$\widetilde{\mathfrak{g}} = \{(x, B) \in \mathfrak{g} \times \mathscr{B} \mid x \in \mathrm{Lie}(B)\}$$

over $\mathscr{B}$. We have a canonical morphism

$$\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g} \qquad \text{given by} \qquad (x, B) \mapsto x,$$

the so-called Grothendieck–Springer simultaneous resolution.

Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone. Let also $\widetilde{\mathcal{N}} = \pi^{-1}(\mathcal{N})$, and let $\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the restriction of π to $\widetilde{\mathcal{N}}$, the so-called Springer resolution. Let $i_{\mathcal{N}} : \mathcal{N} \to \mathfrak{g}$ and $i_{\widetilde{\mathcal{N}}} : \widetilde{\mathcal{N}} \to \widetilde{\mathfrak{g}}$ denote the closed embeddings. Let $\mathfrak{g}_{\mathrm{rs}} \subset \mathfrak{g}$ be the open subvariety consisting of regular semisimple elements, $j_{\mathrm{rs}} : \mathfrak{g}_{\mathrm{rs}} \hookrightarrow \mathfrak{g}$ the inclusion, $\widetilde{\mathfrak{g}}_{\mathrm{rs}} := \pi^{-1}(\mathfrak{g}_{\mathrm{rs}})$, and $\pi_{\mathrm{rs}} : \widetilde{\mathfrak{g}}_{\mathrm{rs}} \to \mathfrak{g}_{\mathrm{rs}}$ the restriction of π.

Fix, once and for all, a G-invariant nondegenerate bilinear form on $\mathfrak{g}$, and set

$$N := (\dim \mathcal{N})/2, \qquad r := \dim \mathfrak{H}, \qquad d := \dim \mathfrak{g} = 2N + r.$$

2.2. Rings and sheaves: conventions and notation. All rings in the paper will be tacitly assumed to be noetherian commutative rings of finite global dimension. If $\Bbbk$ is such a ring, then a '$\Bbbk$-algebra' should be understood to be noetherian, commutative, and of finite global dimension as well, but not necessarily finitely generated over $\Bbbk$. Similarly, all our topological spaces will be tacitly assumed to be locally compact and of finite c-soft dimension in the sense of [KaS, Exercise II.9].

Let $\Bbbk$-mod be the category of $\Bbbk$-modules. If X is a topological space, we write $\mathsf{Sh}(X, \Bbbk)$ for the category of sheaves of $\Bbbk$-modules on X, and $D^{\mathrm{b}}(X, \Bbbk)$ for its bounded derived category. (The objects in these categories are not required to be constructible.) Given $M \in \Bbbk$-mod, let $\underline{M}_X$ (or simply $\underline{M}$) be the associated constant sheaf on X. We regard this operation as an exact functor

$$\underline{(-)}_X : \Bbbk\text{-mod} \to \mathsf{Sh}(X, \Bbbk). \tag{2.1}$$

If $f : X \to Y$ is a continuous map, we have (derived) functors

$$f_*, f_! : D^{\mathrm{b}}(X, \Bbbk) \to D^{\mathrm{b}}(Y, \Bbbk) \qquad \text{and} \qquad f^*, f^! : D^{\mathrm{b}}(Y, \Bbbk) \to D^{\mathrm{b}}(X, \Bbbk).$$

We will denote the nonderived versions of $f_!$ and f^* by ${}^\circ f_!$ and ${}^\circ f^*$, respectively.

If $f : X \to Y$ is a continuous map, there exists an obvious isomorphism of functors which will be denoted by (Const):

$$f^* \circ \underline{(-)}_Y \cong \underline{(-)}_X,$$

If we have continuous maps of topological spaces $X \xrightarrow{f} Y \xrightarrow{g} Z$ and if we let $h = g \circ f$, there are natural isomorphisms which will be denoted (Co):

$$g_! \circ f_! \xrightarrow{\sim} h_! \qquad \text{and} \qquad f^* \circ g^* \xrightarrow{\sim} h^*.$$

Finally, if we have a cartesian square:

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

we will denote by (BC) the base-change isomorphism

$$g^* \circ f_! \xrightarrow{\sim} f'_! \circ g'^*.$$

2.3. Change of rings. If $\Bbbk'$ is a $\Bbbk$-algebra, we have exact forgetful functors

$$\mathrm{For}^{\mathrm{m}} = \mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'} : \Bbbk'\text{-mod} \to \Bbbk\text{-mod} \qquad \text{and} \qquad \mathrm{For}^X_{\Bbbk,\Bbbk'} : \mathsf{Sh}(X,\Bbbk') \to \mathsf{Sh}(X,\Bbbk).$$

The derived functor of the latter will be denoted by the same symbol.

If $f : X \to Y$ is a continuous map of topological spaces, we claim that there is a natural isomorphism

$$\mathrm{For}^Y_{\Bbbk,\Bbbk'} \circ f_! \xrightarrow{\sim} f_! \circ \mathrm{For}^X_{\Bbbk,\Bbbk'}. \tag{2.2}$$

To construct this isomorphism, we first note that the nonderived analogue $\mathrm{For}^Y_{\Bbbk,\Bbbk'} \circ {}^\circ f_! \xrightarrow{\sim} {}^\circ f_! \circ \mathrm{For}^X_{\Bbbk,\Bbbk'}$ is obvious. The left hand side of (2.2) is canonically isomorphic to the right derived functor of $\mathrm{For}^Y_{\Bbbk,\Bbbk'} \circ {}^\circ f_!$. Then, since $\mathrm{For}^X_{\Bbbk,\Bbbk'}$ takes c-soft $\Bbbk'$-sheaves to c-soft $\Bbbk$-sheaves (which are acyclic for $f_!$), the right hand side of (2.2) is canonically isomorphic to the right derived functor of ${}^\circ f_! \circ \mathrm{For}^X_{\Bbbk,\Bbbk'}$ by [KaS, Proposition 1.8.7]. The claim follows. By a similar argument, we also obtain a canonical isomorphism

$$\mathrm{For}^Y_{\Bbbk,\Bbbk'} \circ f^* \xrightarrow{\sim} f^* \circ \mathrm{For}^X_{\Bbbk,\Bbbk'}. \tag{2.3}$$

Finally, we have an obvious isomorphism

$$\mathrm{For}^X_{\Bbbk,\Bbbk'} \circ \underline{(-)}_X \xrightarrow{\sim} \underline{(-)}_X \circ \mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'}. \tag{2.4}$$

Appendix A collects lemmas expressing the compatibility of these isomorphisms with each other and with various other isomorphisms that will be introduced later.

2.4. 'Perverse' sheaves. Let X be a complex algebraic variety, equipped with a stratification by locally closed smooth subvarieties. By [BBD, Corollaire 2.1.4], there is a well-defined 'perverse t-structure' on $D^{\mathrm{b}}(X,\Bbbk)$ (for the middle perversity). Its heart will be denoted $\mathsf{Perv}(X,\Bbbk)$. Not all objects of $\mathsf{Perv}(X,\Bbbk)$ are perverse sheaves in the conventional sense, because they are not required to be constructible. Note that this category depends on the choice of stratification.

In particular, we equip $\mathcal{N}$ with the stratification by G-orbits, and we equip $\mathfrak{g}$ with the Lie algebra analogue of the stratification given in [L2, §3]. In this stratification, the regular semisimple set forms a single stratum. Each G-orbit in $\mathcal{N}$ is also a stratum of $\mathfrak{g}$, so the functor $i_{\mathcal{N}*} : D^{\mathrm{b}}(\mathcal{N},\Bbbk) \to D^{\mathrm{b}}(\mathfrak{g},\Bbbk)$ restricts to an exact functor $i_{\mathcal{N}*} : \mathsf{Perv}(\mathcal{N},\Bbbk) \to \mathsf{Perv}(\mathfrak{g},\Bbbk)$.

2.5. Springer and Grothendieck functors. We define two additive functors:

$$\underline{\mathsf{Spr}}^{\Bbbk} : \Bbbk\text{-mod} \to \mathsf{Perv}(\mathcal{N},\Bbbk) \qquad \text{by} \qquad \underline{\mathsf{Spr}}(M) = \mu_!(\underline{M}_{\widetilde{\mathcal{N}}})[2N],$$

$$\underline{\mathsf{Groth}}^{\Bbbk} : \Bbbk\text{-mod} \to \mathsf{Perv}(\mathfrak{g},\Bbbk) \qquad \text{by} \qquad \underline{\mathsf{Groth}}(M) = \pi_!(\underline{M}_{\widetilde{\mathfrak{g}}})[d].$$

(Sometimes, for simplicity we write $\underline{\mathsf{Spr}}$, $\underline{\mathsf{Groth}}$ instead of $\underline{\mathsf{Spr}}^{\Bbbk}$, $\underline{\mathsf{Groth}}^{\Bbbk}$.) The fact that these functors take values in the heart of the perverse t-structure follows from

the fact that μ is semismall and π is small. These functors are exact functors of abelian categories. For any M in $\Bbbk$-mod we have a canonical isomorphism

$$\underline{\mathsf{Groth}}(M) \cong (j_{\mathrm{rs}})_{!*}\big((\pi_{\mathrm{rs}})_! \underline{M}_{\widetilde{\mathfrak{g}}_{\mathrm{rs}}}[d]\big). \tag{2.5}$$

Lemma 2.1. *The functor $i_{\mathcal{N}}^* \circ \underline{\mathsf{Groth}}[-r] : \Bbbk\text{-mod} \to D^{\mathrm{b}}(\mathcal{N}, \Bbbk)$ takes values in $\mathsf{Perv}(\mathcal{N}, \Bbbk)$. Moreover, there exists a natural isomorphism*

$$i_{\mathcal{N}}^* \circ \underline{\mathsf{Groth}}[-r] \xrightarrow{\sim} \underline{\mathsf{Spr}}. \tag{2.6}$$

Proof. Consider the cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{N}} & \xrightarrow{i_{\widetilde{\mathcal{N}}}} & \widetilde{\mathfrak{g}} \\ {\scriptstyle\mu}\downarrow & \square & \downarrow{\scriptstyle\pi} \\ \mathcal{N} & \xrightarrow{i_{\mathcal{N}}} & \mathfrak{g}. \end{array}$$

Then for any M in $\Bbbk$-mod we have functorial isomorphisms

$$i_{\mathcal{N}}^* \underline{\mathsf{Groth}}(M)[-r] = i_{\mathcal{N}}^* \pi_! (\underline{M}_{\widetilde{\mathfrak{g}}})[2N] \overset{(\mathrm{BC})}{\cong} \mu_! i_{\widetilde{\mathcal{N}}}^* (\underline{M}_{\widetilde{\mathfrak{g}}})[2N] \overset{(\mathrm{Const})}{\cong} \mu_! (\underline{M}_{\widetilde{\mathcal{N}}})[2N],$$

which proves the claim. □

2.6. Springer and Grothendieck functors and change of rings. If $\Bbbk'$ is a $\Bbbk$-algebra, then we also define

$$\underline{\mathsf{Spr}}_{\Bbbk}^{\Bbbk'} = \underline{\mathsf{Spr}}^{\Bbbk} \circ \mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'} : \Bbbk'\text{-mod} \to \mathsf{Perv}(\mathcal{N}, \Bbbk),$$
$$\underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'} = \underline{\mathsf{Groth}}^{\Bbbk} \circ \mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'} : \Bbbk'\text{-mod} \to \mathsf{Perv}(\mathfrak{g}, \Bbbk).$$

Using isomorphisms (2.2) and (2.4) we obtain canonical isomorphisms of functors

$$\underline{\mathsf{Spr}}_{\Bbbk}^{\Bbbk'} \cong \mathrm{For}^{\mathcal{N}}_{\Bbbk,\Bbbk'} \circ \underline{\mathsf{Spr}}^{\Bbbk'}, \qquad \underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'} \cong \mathrm{For}^{\mathfrak{g}}_{\Bbbk,\Bbbk'} \circ \underline{\mathsf{Groth}}^{\Bbbk'}. \tag{2.7}$$

Hence we can define two isomorphisms of functors

$$\phi : i_{\mathcal{N}}^* \circ \underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}[-r] \xrightarrow{\sim} \underline{\mathsf{Spr}}_{\Bbbk}^{\Bbbk'}, \quad \phi' : i_{\mathcal{N}}^* \circ \underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}[-r] \xrightarrow{\sim} \underline{\mathsf{Spr}}_{\Bbbk}^{\Bbbk'} \tag{2.8}$$

in such a way that the following diagrams commute:

$$\begin{array}{ccc} i_{\mathcal{N}}^* \underline{\mathsf{Groth}}^{\Bbbk} \mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'}[-r] & \xrightarrow{(2.6)^{\Bbbk}} & \underline{\mathsf{Spr}}^{\Bbbk} \mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'} \\ \| & & \| \\ i_{\mathcal{N}}^* \underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}[-r] & \xrightarrow{\phi} & \underline{\mathsf{Spr}}_{\Bbbk}^{\Bbbk'} \end{array} \qquad \begin{array}{ccc} \mathrm{For}^{\mathcal{N}}_{\Bbbk,\Bbbk'} i_{\mathcal{N}}^* \underline{\mathsf{Groth}}^{\Bbbk'}[-r] & \xrightarrow{(2.6)^{\Bbbk'}} & \mathrm{For}^{\mathcal{N}}_{\Bbbk,\Bbbk'} \underline{\mathsf{Spr}}^{\Bbbk'} \\ {\scriptstyle (2.3)\&(2.7)}\downarrow & & \downarrow{\scriptstyle (2.7)} \\ i_{\mathcal{N}}^* \underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}[-r] & \xrightarrow{\phi'} & \underline{\mathsf{Spr}}_{\Bbbk}^{\Bbbk'} \end{array}$$

Using Lemma A.2 and Lemma A.3, one can easily check that $\phi = \phi'$. Hence from now on we will use the reference (2.8) to denote either ϕ or ϕ'.

2.7. Fourier–Sato transform. Let Y be a topological space, and $p : E \to Y$ a complex vector bundle. Consider the $\mathbb{C}^\times$-action on the fibers of p given by scaling. Recall that a sheaf on E (resp. an object of $D^{\mathrm{b}}(E, \Bbbk)$) is said to be *conic* if its restrictions to $\mathbb{C}^\times$-orbits are locally constant (resp. its cohomology sheaves are conic). Let $D^{\mathrm{b}}_{\mathrm{con}}(E, \Bbbk) \subset D^{\mathrm{b}}(E, \Bbbk)$ be the full subcategory consisting of conic objects.

Let $\check{p} : E^* \to Y$ be the dual vector bundle. (Here E^* is the dual vector bundle as a *complex* vector bundle. However it can also be regarded as the dual to the *real* vector bundle E via the pairing $E \times E^* \to \mathbb{R}$ sending (x, y) to $\mathrm{Re}(\langle x, y\rangle)$.) Let

$$Q = \{(x, y) \in E \times_Y E^* \mid \mathrm{Re}(\langle x, y\rangle) \leq 0\} \subset E \times_Y E^*.$$

Let $q : Q \to E$ and $\check{q} : Q \to E^*$ be the obvious projection maps. The *Fourier–Sato transform* is defined to be the functor

$$\mathbb{T}_E : D^{\mathrm{b}}_{\mathrm{con}}(E, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{con}}(E^*, \Bbbk) \qquad \text{given by} \qquad \mathbb{T}_E = \check{q}_! q^* [\mathrm{rk}\, E]. \tag{2.9}$$

This functor is an equivalence of categories; see [KaS, Bry]. Note that the shift by $\mathrm{rk}\, E$ does not appear in the definition in [KaS]. Thus, with the notation of *loc. cit.*, we have $\mathbb{T}_E(M) = M^\wedge[\mathrm{rk}\, E]$.

If $\Bbbk'$ is a $\Bbbk$-algebra, from the definition (2.9), we obtain a natural isomorphism

$$\mathrm{For}^{E^*}_{\Bbbk,\Bbbk'} \circ \mathbb{T}_E \xrightarrow{\sim} \mathbb{T}_E \circ \mathrm{For}^{E}_{\Bbbk,\Bbbk'} \tag{2.10}$$

by combining (2.2) and (2.3).

We will use several compatibility properties of the Fourier–Sato transform, which we recall now. (Lemmas asserting the compatibility between the various isomorphisms we introduce are to be found in Appendix A.) Let $y : Y \to E^*$ be the embedding of the zero section. In §A.5 we will construct a natural isomorphism

$$\mathbb{T}_E \circ \underline{(-)}_E \xrightarrow{\sim} y_! \circ \underline{(-)}_Y [-\mathrm{rk}\, E]. \tag{2.11}$$

One of the steps of this construction is an isomorphism proved in Lemma A.7:

$$p_! \underline{(-)}_E \cong \underline{(-)}_Y [-2\,\mathrm{rk}\, E]. \tag{2.12}$$

Let $g : Y' \to Y$ be a continuous map of topological spaces, and form the cartesian squares

$$\begin{array}{ccc} E' = Y' \times_Y E' & \xrightarrow{f} & E \\ \downarrow & \square & \downarrow \\ Y' & \xrightarrow[g]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} (E')^* = Y' \times_Y E^* & \xrightarrow{f'} & E^* \\ \downarrow & \square & \downarrow \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

According to [KaS, Proposition 3.7.13], there is a natural isomorphism

$$\mathbb{T}_E \circ f_! \xrightarrow{\sim} f'_! \circ \mathbb{T}_{E'}. \tag{2.13}$$

Next, let $E_1 \to Y$ and $E_2 \to Y$ be two complex vector bundles, and let $\phi : E_1 \to E_2$ be a map of vector bundles. Let ${}^{\mathrm{t}}\phi : E_2^* \to E_1^*$ denote the dual map. According to [KaS, Proposition 3.7.14], there is a natural isomorphism

$$({}^{\mathrm{t}}\phi)^* \circ \mathbb{T}_{E_1}[-\mathrm{rk}\, E_1] \xrightarrow{\sim} \mathbb{T}_{E_2} \circ \phi_![-\mathrm{rk}\, E_2]. \tag{2.14}$$

A particularly important special case arises when $E_2 = Y$ is the trivial vector bundle over Y, so that ϕ is the bundle projection $p : E_1 = E \to Y$. Then the dual map ${}^{\mathrm{t}}\phi$ is the embedding of the zero section $y : Y \to E^*$, so (2.14) becomes

$$y^* \circ \mathbb{T}_E \xrightarrow{\sim} p_![\mathrm{rk}\, E]. \tag{2.15}$$

Here we have identified the Fourier–Sato transform for the trivial vector bundle with the identity functor.

We will use in particular $\mathbb{T}_E$ in the case $E = \mathfrak{g}$ (considered as a vector bundle over a point). Since $\mathfrak{g}$ is equipped with a nondegenerate bilinear form, we can identify it with its dual, and regard $\mathbb{T}_\mathfrak{g}$ as an autoequivalence $\mathbb{T}_\mathfrak{g} : D^{\mathrm{b}}_{\mathrm{con}}(\mathfrak{g}, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{con}}(\mathfrak{g}, \Bbbk)$.

2.8. Fourier–Sato transform and Springer and Grothendieck functors.

Lemma 2.2. *There is a natural isomorphism of functors*

$$\mathbb{T}_{\mathfrak{g}} \circ \underline{\mathsf{Groth}} \cong i_{\mathcal{N}!} \circ \underline{\mathsf{Spr}}. \tag{2.16}$$

Proof. Consider $\mathfrak{g} \times \mathscr{B}$ as a vector bundle over $\mathscr{B}$. The functor $\mathbb{T}_{\mathfrak{g}\times\mathscr{B}}$ can be regarded as an autoequivalence of $D^{\mathrm{b}}_{\mathrm{con}}(\mathfrak{g} \times \mathscr{B}, \Bbbk)$. Let $x : \widetilde{\mathfrak{g}} \hookrightarrow \mathfrak{g} \times \mathscr{B}$ be the inclusion, which is a morphism of vector bundles over $\mathscr{B}$, and let ${}^{\mathrm{t}}x : \mathfrak{g} \times \mathscr{B} \to (\widetilde{\mathfrak{g}})^*$ be the dual morphism. Let $y : \mathscr{B} \hookrightarrow (\widetilde{\mathfrak{g}})^*$ be the inclusion of the zero section. Finally, let $f : \mathfrak{g} \times \mathscr{B} \to \mathfrak{g}$ be the projection. We observe that the following diagram is cartesian, where z is the natural projection:

$$\begin{array}{ccc} \widetilde{\mathcal{N}} & \xrightarrow{z} & \mathscr{B} \\ {\scriptstyle x\circ i_{\widetilde{\mathcal{N}}}}\downarrow & \square & \downarrow{\scriptstyle y} \\ \mathfrak{g} \times \mathscr{B} & \xrightarrow{{}^{\mathrm{t}}x} & (\widetilde{\mathfrak{g}})^*. \end{array}$$

Now we are in a position to prove the lemma. For M in $\Bbbk$-mod we have

$$\mathbb{T}_{\mathfrak{g}}\underline{\mathsf{Groth}}(M) = \mathbb{T}_{\mathfrak{g}}\pi_!(\underline{M}_{\widetilde{\mathfrak{g}}})[d] \overset{(\mathrm{Co})}{\cong} \mathbb{T}_{\mathfrak{g}} f_! x_!(\underline{M}_{\widetilde{\mathfrak{g}}})[d].$$

We deduce isomorphisms

$$\begin{aligned} \mathbb{T}_{\mathfrak{g}}\underline{\mathsf{Groth}}(M) &\overset{(2.13)}{\cong} f_!\mathbb{T}_{\mathfrak{g}\times\mathscr{B}} x_!(\underline{M}_{\widetilde{\mathfrak{g}}})[d] \overset{(2.14)}{\cong} f_!({}^{\mathrm{t}}x)^*\mathbb{T}_{\widetilde{\mathfrak{g}}}(\underline{M}_{\widetilde{\mathfrak{g}}})[d+N] \\ &\overset{(2.11)}{\cong} f_!({}^{\mathrm{t}}x)^* y_!(\underline{M}_{\mathscr{B}})[2N] \overset{(\mathrm{BC})}{\cong} f_!(x \circ i_{\widetilde{\mathcal{N}}})_! z^*(\underline{M}_{\mathscr{B}})[2N]. \end{aligned}$$

Using the (Const) isomorphism $z^*(\underline{M}_{\mathscr{B}}) \cong \underline{M}_{\widetilde{\mathcal{N}}}$ and the fact that $f \circ x \circ i_{\widetilde{\mathcal{N}}} = i_{\mathcal{N}} \circ \mu$, we deduce the isomorphism of the lemma. □

2.9. Fourier–Sato transform, Springer and Grothendieck functors, and change of rings. Let $\Bbbk'$ be a $\Bbbk$-algebra. Recall the functors $\underline{\mathsf{Groth}}^{\Bbbk'}_{\Bbbk}$ and $\underline{\mathsf{Spr}}^{\Bbbk'}_{\Bbbk}$ of §2.6. As in *loc. cit.* there exist two natural isomorphisms of functors

$$\psi : \mathbb{T}_{\mathfrak{g}} \circ \underline{\mathsf{Groth}}^{\Bbbk'}_{\Bbbk} \xrightarrow{\sim} i_{\mathcal{N}!} \circ \underline{\mathsf{Spr}}^{\Bbbk'}_{\Bbbk}, \quad \psi' : \mathbb{T}_{\mathfrak{g}} \circ \underline{\mathsf{Groth}}^{\Bbbk'}_{\Bbbk} \xrightarrow{\sim} i_{\mathcal{N}!} \circ \underline{\mathsf{Spr}}^{\Bbbk'}_{\Bbbk} \tag{2.17}$$

defined in such a way that the following diagrams commute:

$$\begin{array}{ccc} \mathbb{T}_{\mathfrak{g}}\underline{\mathsf{Groth}}^{\Bbbk}\mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'} & \xrightarrow{(2.16)_{\Bbbk}} & i_{\mathcal{N}!}\underline{\mathsf{Spr}}^{\Bbbk}\mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'} \\ \| & & \| \\ \mathbb{T}_{\mathfrak{g}}\underline{\mathsf{Groth}}^{\Bbbk'}_{\Bbbk} & \xrightarrow{\psi} & i_{\mathcal{N}!}\underline{\mathsf{Spr}}^{\Bbbk'}_{\Bbbk} \end{array} \qquad \begin{array}{ccc} \mathrm{For}^{\mathfrak{g}}_{\Bbbk,\Bbbk'}\mathbb{T}_{\mathfrak{g}}\underline{\mathsf{Groth}}^{\Bbbk'} & \xrightarrow{(2.16)_{\Bbbk'}} & \mathrm{For}^{\mathfrak{g}}_{\Bbbk,\Bbbk'} i_{\mathcal{N}!}\underline{\mathsf{Spr}}^{\Bbbk'} \\ {\scriptstyle (2.10)\&(2.7)}\downarrow & & {\scriptstyle (2.2)\&(2.7)}\downarrow \\ \mathbb{T}_{\mathfrak{g}}\underline{\mathsf{Groth}}^{\Bbbk'}_{\Bbbk} & \xrightarrow{\psi'} & i_{\mathcal{N}!}\underline{\mathsf{Spr}}^{\Bbbk'}_{\Bbbk} \end{array}$$

Using Lemmas A.2, A.4, A.5, and A.6, one can easily check that $\psi = \psi'$. Hence from now on we will use the reference (2.17) to denote either ψ or ψ'.

3. Endomorphisms under change of rings

3.1. Functors commuting with direct sums. Let $\mathscr{C}$ be an abelian category which admits arbitrary direct sums. Recall that a functor $F : \Bbbk\text{-mod} \to \mathscr{C}$ *commutes with direct sums* if for any set I and any collection $(M_i)_{i\in I}$ of $\Bbbk$-modules the following natural morphism is an isomorphism:

$$\bigoplus_{i\in I} F(M_i) \to F\big(\bigoplus_{i\in I} M_i\big). \tag{3.1}$$

Lemma 3.1. *Let $F : \Bbbk\text{-mod} \to \mathscr{C}$ be an exact functor which commutes with direct sums, and let $\mathrm{End}(F)$ be the algebra of endomorphisms of F in the category of functors from $\Bbbk$-mod to $\mathscr{C}$. Then the algebra morphism*

$$\mathrm{ev}_{\Bbbk} : \mathrm{End}(F) \to \mathrm{End}_{\mathscr{C}}(F(\Bbbk))$$

given by evaluation at $\Bbbk$ is an isomorphism.

Proof. First we note that if η is in $\mathrm{End}(F)$, then for any set I and any collection $(M_i)_{i\in I}$ of $\Bbbk$-modules, the following diagram commutes:

$$\begin{array}{ccc} \bigoplus_{i\in I} F(M_i) & \xrightarrow{\oplus_i \eta_{M_i}} & \bigoplus_{i\in I} F(M_i) \\ {\scriptstyle (3.1)}\downarrow\wr & & \wr\downarrow{\scriptstyle (3.1)} \\ F\big(\bigoplus_{i\in I} M_i\big) & \xrightarrow{\eta_{\oplus_i M_i}} & F\big(\bigoplus_{i\in I} M_i\big). \end{array}$$

Let us prove injectivity of our morphism. Assume that $\eta_{\Bbbk} = 0$. Let M be in $\Bbbk$-mod, and choose a set I and a surjection $\Bbbk^{\oplus I} \twoheadrightarrow M$. Then as F is exact we have a surjection $F(\Bbbk^{\oplus I}) \twoheadrightarrow F(M)$. Using the diagram above one checks that $\eta_{\Bbbk^{\oplus I}} = 0$, which implies that $\eta_M = 0$.

Now we prove surjectivity. Let $\zeta : F(\Bbbk) \to F(\Bbbk)$ be a morphism in $\mathscr{C}$. If M is in $\Bbbk$-mod we choose a presentation $\Bbbk^{\oplus J} \xrightarrow{f_1} \Bbbk^{\oplus I} \xrightarrow{f_2} M \to 0$ for some sets I and J. The image under F of this presentation is an exact sequence. Then we define η_M as the unique morphism which makes the following diagram commutative, where we use the canonical isomorphisms $F(\Bbbk^{\oplus I}) \cong F(\Bbbk)^{\oplus I}$ and $F(\Bbbk^{\oplus J}) \cong F(\Bbbk)^{\oplus J}$:

$$\begin{array}{ccccc} F(\Bbbk)^{\oplus J} & \xrightarrow{F(f_1)} & F(\Bbbk)^{\oplus I} & \xrightarrow{F(f_2)} & F(M) \\ {\scriptstyle \zeta^{\oplus J}}\downarrow & & {\scriptstyle \zeta^{\oplus I}}\downarrow & & \downarrow{\scriptstyle \eta_M} \\ F(\Bbbk)^{\oplus J} & \xrightarrow{F(f_1)} & F(\Bbbk)^{\oplus I} & \xrightarrow{F(f_2)} & F(M). \end{array}$$

To check that η_M does not depend on the presentation, and that it defines a morphism of functors, is an easy exercise left to the reader. □

3.2. Endomorphisms. As above we denote by $\mathrm{End}(\underline{\mathsf{Spr}})$ the $\Bbbk$-algebra of endomorphisms of the functor $\underline{\mathsf{Spr}}$, and likewise for $\mathrm{End}(\underline{\mathsf{Groth}})$.

Lemma 3.2. *The following $\Bbbk$-algebra homomorphisms are isomorphisms:*

$$\mathrm{End}(\underline{\mathsf{Spr}}) \xrightarrow{\mathrm{ev}_{\Bbbk}} \mathrm{End}_{\mathsf{Perv}(\mathcal{N},\Bbbk)}(\underline{\mathsf{Spr}}(\Bbbk)), \tag{3.2}$$

$$\mathrm{End}(\underline{\mathsf{Groth}}) \xrightarrow{\mathrm{ev}_{\Bbbk}} \mathrm{End}_{\mathsf{Perv}(\mathfrak{g},\Bbbk)}(\underline{\mathsf{Groth}}(\Bbbk)) \xrightarrow{j_{\mathrm{rs}}^*} \mathrm{End}_{\mathsf{Perv}(\mathfrak{g}_{\mathrm{rs}},\Bbbk)}(\underline{\mathsf{Groth}}(\Bbbk)|_{\mathfrak{g}_{\mathrm{rs}}}). \tag{3.3}$$

Proof. The functors $\underline{\mathsf{Groth}}$ and $\underline{\mathsf{Spr}}$ both commute with direct sums: this follows from the observation that the functor (2.1) commutes with direct sums, as well as the functors $\pi_!$ and $\mu_!$. Using Lemma 3.1, we deduce that (3.2) and the first morphism of (3.3) are isomorphisms. The fact that the second morphism of (3.3) is also an isomorphism follows from (2.5) and the well-known property that the functor $(j_{\mathrm{rs}})_{!*}$ is fully faithful (see e.g. [Ju2, Proposition 2.29]). □

3.3. Change of rings. The next two propositions deal with a comparison between two coefficient rings. Let $\Bbbk'$ be a $\Bbbk$-algebra. Recall the functors $\underline{\mathsf{Spr}}_{\Bbbk}^{\Bbbk'}$ and $\underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}$ introduced in §2.6. Composing on the right with $\mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'}$, respectively composing on the left with $\mathrm{For}^{\mathfrak{g}}_{\Bbbk,\Bbbk'}$ and using isomorphism (2.7), we obtain natural maps

$$\mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk}) \to \mathrm{End}(\underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}) \leftarrow \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}),$$

and likewise for $\underline{\mathsf{Spr}}$. The map $\mathrm{End}(\underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}) \to \mathrm{End}(i_{\mathcal{N}}^* \circ \underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'})$ induced by $i_{\mathcal{N}}^*$, combined with isomorphism (2.8), gives a morphism of $\Bbbk$-algebras

$$\tilde{\rho}_{\Bbbk}^{\Bbbk'} : \mathrm{End}(\underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}) \to \mathrm{End}(\underline{\mathsf{Spr}}_{\Bbbk}^{\Bbbk'}).$$

Similar constructions using isomorphism (2.6) provide $\Bbbk$-algebra morphisms

$$\tilde{\rho}^{\Bbbk} : \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk}) \to \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk}) \qquad \text{and} \qquad \tilde{\rho}^{\Bbbk'} : \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}) \to \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'}).$$

Similarly, the map $\mathrm{End}(\underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}) \to \mathrm{End}(\mathbb{T}_{\mathfrak{g}} \circ \underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'})$ induced by $\mathbb{T}_{\mathfrak{g}}$, combined with isomorphisms (2.16) and (2.17) and the fact that $i_{\mathcal{N}!}$ is fully faithful, gives rise to morphisms of $\Bbbk$-algebras

$$\tilde{\varphi}_{\Bbbk}^{\Bbbk'} : \mathrm{End}(\underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}) \to \mathrm{End}(\underline{\mathsf{Spr}}_{\Bbbk}^{\Bbbk'}),$$
$$\tilde{\varphi}^{\Bbbk} : \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk}) \to \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk}), \qquad \tilde{\varphi}^{\Bbbk'} : \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}) \to \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'}).$$

Since $\mathbb{T}_{\mathfrak{g}}$ is an equivalence of categories, these morphisms are algebra isomorphisms.

Lemma 3.3. *The following diagrams commute, where the vertical arrows are defined above:*

$$\begin{array}{ccc}
\mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk}) & \xrightarrow{\tilde{\rho}^{\Bbbk}} & \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk}) \\
\downarrow & & \downarrow \\
\mathrm{End}(\underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}) & \xrightarrow{\tilde{\rho}_{\Bbbk}^{\Bbbk'}} & \mathrm{End}(\underline{\mathsf{Spr}}_{\Bbbk}^{\Bbbk'}) \\
\uparrow & & \uparrow \\
\mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}) & \xrightarrow{\tilde{\rho}^{\Bbbk'}} & \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'})
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk}) & \xrightarrow{\tilde{\varphi}^{\Bbbk}} & \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk}) \\
\downarrow & & \downarrow \\
\mathrm{End}(\underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}) & \xrightarrow{\tilde{\varphi}_{\Bbbk}^{\Bbbk'}} & \mathrm{End}(\underline{\mathsf{Spr}}_{\Bbbk}^{\Bbbk'}) \\
\uparrow & & \uparrow \\
\mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}) & \xrightarrow{\tilde{\varphi}^{\Bbbk'}} & \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'})
\end{array}$$

Proof. Consider the left-hand diagram. The commutativity of the upper square easily follows from the definition of (2.8) as the morphism ϕ of §2.6, and the commutativity of the lower square easily follows from the equivalent definition of this morphism as the morphism ϕ' of §2.6. The proof of the commutativity of the right-hand diagram is similar, using the morphisms ψ and ψ' of §2.9. □

Lemma 3.4. *The maps*

$$\mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}) \to \mathrm{End}(\underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}) \qquad \textit{and} \qquad \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'}) \to \mathrm{End}(\underline{\mathsf{Spr}}_{\Bbbk}^{\Bbbk'})$$

defined above are injective.

Proof. For the first assertion, by Lemma 3.2, it suffices to show that the map

$$\mathrm{End}_{\mathsf{Perv}(\mathfrak{g}_{\mathrm{rs}},\Bbbk')}(\underline{\mathsf{Groth}}^{\Bbbk'}(\Bbbk')|_{\mathfrak{g}_{\mathrm{rs}}}) \to \mathrm{End}_{\mathsf{Perv}(\mathfrak{g}_{\mathrm{rs}},\Bbbk)}(\underline{\mathsf{Groth}}_{\Bbbk}^{\Bbbk'}(\Bbbk')|_{\mathfrak{g}_{\mathrm{rs}}})$$

is injective. But this is clear, since we are now comparing two endomorphism rings of a single locally constant sheaf on $\mathfrak{g}_{\mathrm{rs}}$. The result for $\underline{\mathsf{Spr}}$ then follows from the second diagram in Lemma 3.3, using the fact that the morphisms $\tilde{\varphi}$ are isomorphisms. □

3.4. W-action on $\underline{\mathsf{Groth}}$. Let us now construct a canonical isomorphism

$$r_{\Bbbk} : \Bbbk[\mathbf{W}] \xrightarrow{\sim} \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk}).$$

Throughout this construction, the notation $\underline{M}$ will indicate a constant sheaf on $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$. Because π_{rs} is a regular covering map whose group of deck transformations is $\mathbf{W}$, there is a natural action of $\mathbf{W}$ on any object of the form $\pi_{\mathrm{rs}*}\underline{M}$. Since all our objects live in $\Bbbk$-linear categories, this extends to a natural action of the ring $\Bbbk[\mathbf{W}]$. Moreover, the morphism $\tilde{r}_{\Bbbk} : \Bbbk[\mathbf{W}] \to \mathrm{End}_{\mathsf{Sh}(\mathfrak{g}_{\mathrm{rs}},\Bbbk)}(\pi_{\mathrm{rs}*}\underline{\Bbbk})$ induced by this action is an isomorphism. Combining this with Lemma 3.2 provides the isomorphism $r_{\Bbbk}$.

Proposition 3.5. *If $\Bbbk'$ is a $\Bbbk$-algebra, the morphism $\mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk}) \to \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}_{\Bbbk})$ defined in §3.3 factors (uniquely) through a morphism*

$$\alpha^{\Bbbk'}_{\Bbbk} : \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk}) \to \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}).$$

Moreover the following diagram commutes, where the left vertical morphism is the natural algebra morphism:

$$\begin{array}{ccc} \Bbbk[\mathbf{W}] & \xrightarrow{r_{\Bbbk}} & \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk}) \\ \downarrow & & \downarrow{\scriptstyle \alpha^{\Bbbk'}_{\Bbbk}} \\ \Bbbk'[\mathbf{W}] & \xrightarrow[r_{\Bbbk'}]{} & \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}) \end{array}$$

Proof. Consider the following diagram:

$$\begin{array}{ccccccc} \Bbbk[\mathbf{W}] & \xrightarrow[\sim]{\tilde{r}_{\Bbbk}} & \mathrm{End}_{\mathsf{Sh}(\mathfrak{g}_{\mathrm{rs}},\Bbbk)}(\pi_{\mathrm{rs}*}\underline{\Bbbk}) & \xleftarrow[\sim]{c} & \mathrm{End}_{\mathsf{Perv}(\mathfrak{g},\Bbbk)}(\underline{\mathsf{Groth}}^{\Bbbk}(\Bbbk)) & \xleftarrow[\sim]{\mathrm{ev}_{\Bbbk}} & \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk}) \\ \downarrow & & & & & \swarrow{\scriptstyle \mathrm{ev}_{\Bbbk'}} & \downarrow{\scriptstyle a} \\ & & \mathrm{End}_{\mathsf{Sh}(\mathfrak{g}_{\mathrm{rs}},\Bbbk)}(\pi_{\mathrm{rs}*}\underline{\Bbbk}') & \xleftarrow[\sim]{c} & \mathrm{End}_{\mathsf{Perv}(\mathfrak{g},\Bbbk)}(\underline{\mathsf{Groth}}^{\Bbbk}(\Bbbk')) & \xleftarrow[\sim]{\mathrm{ev}_{\Bbbk'}} & \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}_{\Bbbk}) \\ & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} \\ \Bbbk'[\mathbf{W}] & \xrightarrow[\sim]{\tilde{r}_{\Bbbk'}} & \mathrm{End}_{\mathsf{Sh}(\mathfrak{g}_{\mathrm{rs}},\Bbbk')}(\pi_{\mathrm{rs}*}\underline{\Bbbk}') & \xleftarrow[\sim]{c} & \mathrm{End}_{\mathsf{Perv}(\mathfrak{g},\Bbbk')}(\underline{\mathsf{Groth}}^{\Bbbk'}(\Bbbk')) & \xleftarrow[\sim]{\mathrm{ev}_{\Bbbk'}} & \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}) \end{array}$$

The map labelled "a" is induced by the functor $\mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'}$, the maps "$b$" are induced by $\mathrm{For}^{\mathfrak{g}}_{\Bbbk,\Bbbk'}$ or $\mathrm{For}^{\mathfrak{g}_{\mathrm{rs}}}_{\Bbbk,\Bbbk'}$, and the maps "$c$" are induced by j^*_{rs}. By Lemma 3.1, the horizontal maps from the fourth column to the third one are isomorphisms.

The two squares on the bottom of the diagram are obviously commutative, as is the triangle in the top right corner. Hence the diagram defines two maps from $\Bbbk[\mathbf{W}]$ to $\mathrm{End}_{\mathsf{Sh}(\mathfrak{g}_{\mathrm{rs}},\Bbbk)}(\pi_{\mathrm{rs}*}\underline{\Bbbk}')$. We claim that these maps coincide: this follows from the fact that these maps are induced by the $\mathbf{W}$-action on $\pi_{\mathrm{rs}*}\underline{\Bbbk}'$, which is independent of the coefficient ring. Hence the diagram as a whole is commutative.

It follows that the image of $\mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk}) \xrightarrow{a} \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}_{\Bbbk})$ is contained in the image of $\mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}) \xrightarrow{b} \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}_{\Bbbk})$. Since the latter map is injective (see Lemma 3.4), the existence of $\alpha^{\Bbbk'}_{\Bbbk}$ follows. The commutativity of the diagram is clear. □

3.5. W-actions on $\underline{\mathsf{Spr}}$. We now define

$$\rho^{\Bbbk} : \Bbbk[\mathbf{W}] \to \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk}) \qquad \text{by} \qquad \rho^{\Bbbk} = \tilde{\rho}^{\Bbbk} \circ r_{\Bbbk},$$
$$\varphi^{\Bbbk} : \Bbbk[\mathbf{W}] \to \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk}) \qquad \text{by} \qquad \varphi^{\Bbbk} = \tilde{\varphi}^{\Bbbk} \circ r_{\Bbbk}.$$

Like $\tilde{\varphi}^{\Bbbk}$, the map $\varphi^{\Bbbk}$ is an isomorphism of $\Bbbk$-algebras. By composing with the isomorphism in Lemma 3.2, we obtain the maps $\dot{\rho}^{\Bbbk}, \dot{\varphi}^{\Bbbk}$ considered in Theorem 1.1.

Proposition 3.6. *If $\Bbbk'$ is a $\Bbbk$-algebra, there is a canonical $\Bbbk$-algebra homomorphism*

$$\beta_{\Bbbk}^{\Bbbk'} : \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk}) \to \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'})$$

such that the following diagrams commute:

$$\begin{array}{ccc} \Bbbk[\mathbf{W}] & \xrightarrow{\rho^{\Bbbk}} & \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk}) \\ \downarrow & & \downarrow{\scriptstyle \beta_{\Bbbk}^{\Bbbk'}} \\ \Bbbk'[\mathbf{W}] & \xrightarrow{\rho^{\Bbbk'}} & \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'}) \end{array} \qquad \begin{array}{ccc} \Bbbk[\mathbf{W}] & \xrightarrow{\varphi^{\Bbbk}} & \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk}) \\ \downarrow & & \downarrow{\scriptstyle \beta_{\Bbbk}^{\Bbbk'}} \\ \Bbbk'[\mathbf{W}] & \xrightarrow{\varphi^{\Bbbk'}} & \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'}) \end{array}$$

Proof. Recall the morphisms considered in Lemma 3.3. We claim first that the image of the morphism $\mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk}) \to \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'}_{\Bbbk})$ is contained in the image of the map $\mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'}) \to \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'}_{\Bbbk})$. Indeed, this follows from the corresponding assertion for $\underline{\mathsf{Groth}}$ (proved in the course of establishing Proposition 3.5) and the second commutative diagram in Lemma 3.3, in which the horizontal maps are isomorphisms. In view of Lemma 3.4, we get a natural map $\beta_{\Bbbk}^{\Bbbk'} : \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk}) \to \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'})$, and then the two commutative diagrams in Lemma 3.3 yield the simpler diagrams

$$\begin{array}{ccc} \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk}) & \xrightarrow{\tilde{\rho}^{\Bbbk}} & \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk}) \\ {\scriptstyle \alpha_{\Bbbk}^{\Bbbk'}}\downarrow & & \downarrow{\scriptstyle \beta_{\Bbbk}^{\Bbbk'}} \\ \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}) & \xrightarrow{\tilde{\rho}^{\Bbbk'}} & \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk}) & \xrightarrow{\tilde{\varphi}^{\Bbbk}} & \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk}) \\ {\scriptstyle \alpha_{\Bbbk}^{\Bbbk'}}\downarrow & & \downarrow{\scriptstyle \beta_{\Bbbk}^{\Bbbk'}} \\ \mathrm{End}(\underline{\mathsf{Groth}}^{\Bbbk'}) & \xrightarrow{\tilde{\varphi}^{\Bbbk'}} & \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk'}) \end{array}$$

We conclude by combining these diagrams with Proposition 3.5. □

3.6. Comparing the actions. We can now explain how the main result of the paper, Theorem 1.1, reduces to the $\Bbbk = \mathbb{Z}$ case. By Lemma 3.2, Theorem 1.1 (for any given $\Bbbk$) is equivalent to the following statement.

Theorem 3.7. *The two maps $\rho^{\Bbbk}, \varphi^{\Bbbk} : \Bbbk[\mathbf{W}] \to \mathrm{End}(\underline{\mathsf{Spr}}^{\Bbbk})$ are related as follows:*

$$\rho^{\Bbbk} = \varphi^{\Bbbk} \circ \mathsf{sgn}.$$

Since we can regard $\Bbbk$ as a $\mathbb{Z}$-algebra, it is obvious from Proposition 3.6 that it suffices to prove Theorem 3.7 in the $\Bbbk = \mathbb{Z}$ case. The $\Bbbk = \mathbb{Z}$ case of Theorem 1.1 is covered by known arguments, explained in the next section.

4. Proof of the 'easy' case

Here we explain how to prove Theorem 1.1 in the 'easy' case of §1.3. The required arguments appeared first around thirty years ago in papers by Hotta [Ho], Brylinski [Bry] and Spaltenstein [Spa], and subsequently in the surveys by Shoji [Sh] and Jantzen [Ja2]. Those references considered the étale topology and ℓ-adic cohomology, so some adaptations to our setting are needed. The other reason for revisiting these arguments is that our statement is slightly more precise than, for example, [Sh, Proposition 17.7], in that it amounts to proving $\mathbf{W}$-equivariance of a specific isomorphism rather than the mere existence of a $\mathbf{W}$-equivariant isomorphism.

4.1. Actions on the cohomology of $\mathscr{B}$. We begin by recalling the classical $\mathbf{W}$-action on $H^\bullet(\mathscr{B};\Bbbk)$. Choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$; we obtain a canonical identification $\mathbf{W} = N_G(T)/T$. Then $\mathbf{W}$ acts on G/T and on $H^\bullet(G/T;\Bbbk)$. The map $G/T \to G/B = \mathscr{B}$ is a fibre bundle whose fibre B/T is an affine space of dimension N, and it thus induces an isomorphism

$$H^\bullet(\mathscr{B};\Bbbk) \cong H^\bullet(G/T;\Bbbk). \tag{4.1}$$

We use this isomorphism to define a $\mathbf{W}$-action on $H^\bullet(\mathscr{B};\Bbbk)$. Note that this construction is compatible with change of rings in the obvious sense: if $\Bbbk'$ is a $\Bbbk$-algebra then the natural morphism $H^\bullet(\mathscr{B};\Bbbk) \to H^\bullet(\mathscr{B};\Bbbk')$ is $\mathbf{W}$-equivariant. A similar construction defines a $\mathbf{W}$-action on the Chow ring $A(\mathscr{B})$ of $\mathscr{B}$ (see [De2, §4.7]).

Lemma 4.1. *There is a canonical isomorphism $\Bbbk \otimes_{\mathbb{Z}} A(\mathscr{B}) \cong H^\bullet(\mathscr{B};\Bbbk)$ of graded $\Bbbk[\mathbf{W}]$-modules.*

Proof. Since $\mathscr{B}$ is paved by affine spaces, $H^\bullet(\mathscr{B};\Bbbk)$ is a free $\Bbbk$-module of rank $|\mathbf{W}|$, and the natural morphism $H^\bullet(\mathscr{B};\mathbb{Z}) \to H^\bullet(\mathscr{B};\Bbbk)$ induces an isomorphism $\Bbbk \otimes_{\mathbb{Z}} H^\bullet(\mathscr{B};\mathbb{Z}) \xrightarrow{\sim} H^\bullet(\mathscr{B};\Bbbk)$. Hence it is enough to prove the lemma when $\Bbbk = \mathbb{Z}$.

There is a canonical morphism $A(\mathscr{B}) \to H^\bullet(\mathscr{B};\mathbb{Z})$, see e.g. [Fu, §19.1]. By functoriality this morphism is $\mathbf{W}$-equivariant. The arguments in [Ja2, Proposition 13.1] prove that it is injective. Surjectivity follows from the fact that $H^\bullet(\mathscr{B};\mathbb{Z})$ is spanned by characteristic classes of Schubert varieties. (See also [De1, §8].) □

Lemma 4.2. *Let $\mathbb{F}$ be either $\mathbb{Q}$ or $\mathbb{F}_p$, where p does not divide $|\mathbf{W}|$. Then the $\mathbb{F}[\mathbf{W}]$-module $\mathbb{F} \otimes_{\mathbb{Z}} A(\mathscr{B})$ is isomorphic to the regular representation.*

Proof. By [De2, §4.6], $\mathbb{Q} \otimes_{\mathbb{Z}} A(\mathscr{B})$ is isomorphic to the coinvariant algebra $C_{\mathbb{Q}}$. Hence when $\mathbb{F} = \mathbb{Q}$ the result follows from [Bou, Ch. V, §5, Théorème 2]. If $\mathbb{F} = \mathbb{F}_p$, recall that the composition factors of the modular reduction of a $\mathbb{Q}_p[\mathbf{W}]$-module do not depend on the $\mathbb{Z}_p$-lattice used for the reduction, see [Se, §15.2]. As $\mathbb{Q}_p \otimes_{\mathbb{Z}} A(\mathscr{B})$ is isomorphic to the regular representation, we deduce that $\mathbb{F} \otimes_{\mathbb{Z}} A(\mathscr{B})$ has the same composition factors as $\mathbb{F}[\mathbf{W}]$. Under our assumptions $\mathbb{F}[\mathbf{W}]$ is a semisimple ring, whence the claim. □

We denote by $\Bbbk_{\mathrm{triv}}$, respectively $\Bbbk_\varepsilon$, the $\Bbbk[\mathbf{W}]$-module which is free of rank one over $\Bbbk$ with trivial $\mathbf{W}$-action, respectively with $\mathbf{W}$-action defined by ε.

Lemma 4.3. (1) *The $\Bbbk[\mathbf{W}]$-module $H^{2N}(\mathscr{B};\Bbbk)$ is isomorphic to $\Bbbk_\varepsilon$.*

(2) *Assume that $\Bbbk$ embeds either in a $\mathbb{Q}$-algebra, or in an $\mathbb{F}_p$-algebra where p does not divide $|\mathbf{W}|$. Then the $\Bbbk[\mathbf{W}]$-module $H^\bullet(\mathscr{B};\Bbbk)$ is faithful.*

Proof. (1) By Lemma 4.1, it is enough to prove the analogous claim for $A(\mathscr{B})$. We already know that $A^{2N}(\mathscr{B})$ is a free $\mathbb{Z}$-module of rank 1, and we only have to identify the $\mathbf{W}$-action. However, the action of simple reflections is described in [De2, §4.7], which is sufficient to prove the claim.

(2) It is enough to prove the claim when $\Bbbk$ is a $\mathbb{Q}$-algebra, or an $\mathbb{F}_p$-algebra where p does not divide $|\mathbf{W}|$. However in this case, by Lemmas 4.1 and 4.2, the $\Bbbk[\mathbf{W}]$-module $H^\bullet(\mathscr{B};\Bbbk)$ is isomorphic to the regular representation, which is faithful. □

One can also consider cohomology with compact support. As above there is a canonical isomorphism

$$H_c^\bullet(\mathscr{B};\Bbbk) \cong H_c^{\bullet+2N}(G/T;\Bbbk),$$

which we can use to define a $\mathbf{W}$-action on $H_c^\bullet(\mathscr{B};\Bbbk)$.

Lemma 4.4. *The* $\mathbf{W}$*-module* $H_c^{2N}(\mathscr{B};\Bbbk)$ *is isomorphic to* $\Bbbk_{\mathrm{triv}}$.

Proof. By definition we have an isomorphism $H_c^{2N}(\mathscr{B};\Bbbk) \cong H_c^{4N}(G/T;\Bbbk)$ of $\mathbf{W}$-modules. Now $H_c^{4N}(G/T;\Bbbk)$ is free of rank one, with a canonical generator given by the orientation of G/T induced by the complex structure. This generator is fixed by $\mathbf{W}$, which proves the claim. □

Of course, since $\mathscr{B}$ is compact, one would not normally distinguish $H_c^\bullet(\mathscr{B};\Bbbk)$ from $H^\bullet(\mathscr{B};\Bbbk)$. We do so here because the $\mathbf{W}$-actions are not quite the same:

Lemma 4.5. *The natural isomorphism*

$$H_c^\bullet(\mathscr{B};\Bbbk) \xrightarrow{\sim} H^\bullet(\mathscr{B};\Bbbk) \tag{4.2}$$

is $(\mathbf{W},\varepsilon)$*-equivariant, in the sense that it becomes* $\mathbf{W}$*-equivariant when one of the two sides is tensored with the sign representation of* $\mathbf{W}$.

Proof. For any i we have natural (in particular, $\mathbf{W}$-equivariant) pairings giving the horizontal arrows of the commutative diagram

$$\begin{array}{ccc} H^i(\mathscr{B};\Bbbk) \otimes_\Bbbk H_c^{2N-i}(\mathscr{B};\Bbbk) & \longrightarrow & H_c^{2N}(\mathscr{B};\Bbbk) \\ \downarrow & & \downarrow \\ H^i(\mathscr{B};\Bbbk) \otimes_\Bbbk H^{2N-i}(\mathscr{B};\Bbbk) & \longrightarrow & H^{2N}(\mathscr{B};\Bbbk) \end{array}$$

where the vertical arrows are given by (4.2). By Poincaré duality, the upper arrow is a perfect pairing. (Note that $H^\bullet(\mathscr{B};\Bbbk)$ is free, so that Poincaré duality holds with coefficients in $\Bbbk$.) Hence the same is true for the lower arrow. By Lemma 4.3 and Lemma 4.4, the right vertical map is $(\mathbf{W},\varepsilon)$-equivariant. The claim follows. □

4.2. Comparison with Springer actions. For any variety X, let a_X denote the constant map $X \to \mathrm{pt}$. Factoring $a_{\widetilde{\mathfrak{g}}}$ as either $a_{\mathfrak{g}} \circ \pi$ or as $a_{\mathscr{B}} \circ t$ (where $t : \widetilde{\mathfrak{g}} \to \mathscr{B}$ is the projection, a vector bundle of rank $N + r$), we see that there are isomorphisms

$$H^\bullet\big(a_{\mathfrak{g}!}\underline{\mathsf{Groth}}(\Bbbk)\big) \xrightarrow{\sim} H^\bullet\big(a_{\widetilde{\mathfrak{g}}!}\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[d]\big) \xrightarrow{\sim} H_c^{\bullet-r}(\mathscr{B};\Bbbk). \tag{4.3}$$

On the other hand, applying base-change to the cartesian square

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{j_{\mathscr{B}}} & \widetilde{\mathfrak{g}} \\ \downarrow & \square & \downarrow \pi \\ \{0\} & \xrightarrow[j_0]{} & \mathfrak{g} \end{array}$$

(where all morphisms are the natural ones) we obtain an isomorphism

$$H^\bullet\big(j_0^*\underline{\mathsf{Groth}}(\Bbbk)\big) \xrightarrow{\sim} H^{\bullet+d}(\mathscr{B};\Bbbk). \tag{4.4}$$

Let us denote by $\dot{r}_\Bbbk$ the composition of $r_\Bbbk$ with the morphism $\mathrm{ev}_\Bbbk : \mathrm{End}(\underline{\mathsf{Groth}}) \to \mathrm{End}(\underline{\mathsf{Groth}}(\Bbbk))$. Both (4.3) and (4.4) are $\mathbf{W}$-equivariant, in the following sense.

Lemma 4.6. (1) *Let* $\gamma : \mathrm{End}(\underline{\mathsf{Groth}}(\Bbbk)) \to \mathrm{End}(H^\bullet(\mathscr{B};\Bbbk))$ *be the map induced by* (4.4). *Then* $\gamma \circ \dot{r}_\Bbbk : \Bbbk[\mathbf{W}] \to \mathrm{End}(H^\bullet(\mathscr{B};\Bbbk))$ *coincides with the* $\mathbf{W}$*-action on* $H^\bullet(\mathscr{B};\Bbbk)$ *defined in* §4.1.

(2) *Let* $\gamma_c : \mathrm{End}(\underline{\mathsf{Groth}}(\Bbbk)) \to \mathrm{End}(H_c^\bullet(\mathscr{B};\Bbbk))$ *be the map induced by* (4.3). *Then* $\gamma_c \circ \dot{r}_\Bbbk : \Bbbk[\mathbf{W}] \to \mathrm{End}(H_c^\bullet(\mathscr{B};\Bbbk))$ *coincides with the* $\mathbf{W}$*-action on* $H_c^\bullet(\mathscr{B};\Bbbk)$ *defined in* §4.1.

Proof. Spaltenstein's proof of the ℓ-adic analogue of (1) [Spa, Theorem 2], which is recapitulated in [Sh, Proposition 5.4] and [Ja2, Proposition 13.7], adapts directly to our setting by simply replacing ℓ-adic cohomology with $\Bbbk$-cohomology throughout. A large part of this adapted proof amounts to showing that the following variant of (4.3) is $\mathbf{W}$-equivariant: $H^\bullet\big(a_{\mathfrak{g}*}\underline{\mathsf{Groth}}(\Bbbk)\big) \xrightarrow{\sim} H^\bullet\big(a_{\widetilde{\mathfrak{g}}*}\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[d]\big) \xrightarrow{\sim} H^{\bullet+d}(\mathscr{B};\Bbbk)$. The proof of (2) is entirely analogous, but with $H_c^{\bullet+2\dim X}(X;\Bbbk)$ instead of $H^\bullet(X;\Bbbk)$ and with induced maps on cohomology going in the reverse direction. □

4.3. Proof of the theorem. We consider the cartesian diagram

$$\begin{array}{ccccc}
\mathscr{B} & \xrightarrow{i_{\mathscr{B}}} & \widetilde{\mathcal{N}} & \xrightarrow{i_{\widetilde{\mathcal{N}}}} & \widetilde{\mathfrak{g}} \\
\downarrow & \square & \downarrow{\scriptstyle\mu} & \square & \downarrow{\scriptstyle\pi} \\
\{0\} & \xrightarrow[i_0]{} & \mathcal{N} & \xrightarrow[i_{\mathcal{N}}]{} & \mathfrak{g}
\end{array}$$

where again all maps are the natural ones. By the base-change theorem, we obtain an isomorphism

$$H^{\bullet-2N}\big(i_0^*\underline{\mathsf{Spr}}(\Bbbk)\big) \xrightarrow{\sim} H^\bullet(\mathscr{B};\Bbbk).$$

This induces a map of endomorphism rings, which we write as

$$\sigma : \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk)) \to \mathrm{End}(H^\bullet(\mathscr{B};\Bbbk)) \overset{(4.2)}{\cong} \mathrm{End}(H_c^\bullet(\mathscr{B};\Bbbk)). \tag{4.5}$$

Denote by $\tilde{\rho}(\Bbbk), \tilde{\varphi}(\Bbbk) : \mathrm{End}(\underline{\mathsf{Groth}}(\Bbbk)) \to \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk))$ the maps induced by $\tilde{\rho}^{\Bbbk}$, $\tilde{\varphi}^{\Bbbk}$.

Lemma 4.7. *We have $\gamma = \sigma \circ \tilde{\rho}(\Bbbk)$ and $\gamma_c = \sigma \circ \tilde{\varphi}(\Bbbk)$.*

Proof. The first equality follows from compatibility of the base-change isomorphism with the composition isomorphism for $(\cdot)^*$ functors, see [AHR, Lemma B.7(a)]. The second equality follows from the commutativity of the following diagram:

$$\begin{array}{ccccc}
j_0^*\mathbb{T}_{\mathfrak{g}}\underline{\mathsf{Groth}}(\Bbbk) & \xrightarrow{(2.15)} & a_{\mathfrak{g}!}\underline{\mathsf{Groth}}(\Bbbk)[d] & \xrightarrow{(\mathrm{Co})} & a_{\mathscr{B}!}t_!\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[2d] \\
{\scriptstyle(2.16)}\downarrow & & & & \downarrow{\scriptstyle(2.12)} \\
j_0^*i_{\mathcal{N}!}\underline{\mathsf{Spr}}(\Bbbk) & \xrightarrow{(\mathrm{BC})} & i_0^*\underline{\mathsf{Spr}}(\Bbbk) & \xrightarrow{(\mathrm{BC})} & a_{\mathscr{B}!}\underline{\Bbbk}_{\mathscr{B}}[2N]
\end{array}$$

To prove this commutativity, one must analyse the left vertical map, obtained by applying the functor j_0^* to the isomorphism (2.16). That is, one must recall from the proof of Lemma 2.2 the individual isomorphisms of which (2.16) is the composition, and determine what isomorphism each of these induces on the stalk at 0. Apart from basic compatibilities between composition and base-change isomorphisms as in [AHR, Lemmas B.4 and B.7], the key results are Lemmas A.8, A.9 and A.10. Explicitly, a special case of Lemma A.9 provides the commutativity of the diagram

$$\begin{array}{ccccc}
j_0^*\mathbb{T}_{\mathfrak{g}}f_!x_!\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[d] & \xrightarrow{(2.15)} & a_{\mathfrak{g}!}f_!x_!\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[2d] & \xrightarrow{(\mathrm{Co})} & (a_{\mathfrak{g}\times\mathscr{B}})_!x_!\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[2d] \\
{\scriptstyle(2.13)}\downarrow & & & & \downarrow{\scriptstyle(\mathrm{Co})} \\
j_0^*f_!\mathbb{T}_{\mathfrak{g}\times\mathscr{B}}x_!\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[d] & \xrightarrow{(\mathrm{BC})} & a_{\mathscr{B}!}(y')^*\mathbb{T}_{\mathfrak{g}\times\mathscr{B}}x_!\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[d] & \xrightarrow{(2.15)} & a_{\mathscr{B}!}p'_!x_!\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[2d]
\end{array}$$

where p' denotes the vector bundle projection $\mathfrak{g}\times\mathscr{B} \to \mathscr{B}$ and $y' : \mathscr{B} \to \mathfrak{g}\times\mathscr{B}$ denotes the embedding of the zero section in that vector bundle. Similarly,

Lemma A.10 provides the commutativity of the diagram

$$\begin{array}{ccccc} a_{\mathscr{B}!}(y')^*\mathbb{T}_{\mathfrak{g}\times\mathscr{B}}x_!\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[d] & & \xrightarrow{\quad(2.15)\quad} & & a_{\mathscr{B}!}p'_!x_!\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[2d] \\ \downarrow{\scriptstyle(2.14)} & & & & \downarrow{\scriptstyle(\mathrm{Co})} \\ a_{\mathscr{B}!}(y')^*({}^{\mathrm{t}}x)^*\mathbb{T}_{\widetilde{\mathfrak{g}}}\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[d+N] & \xrightarrow{(\mathrm{Co})} & a_{\mathscr{B}!}y^*\mathbb{T}_{\widetilde{\mathfrak{g}}}\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[d+N] & \xrightarrow{(2.15)} & a_{\mathscr{B}!}t_!\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[2d] \end{array}$$

and Lemma A.8 provides the commutativity of the diagram

$$\begin{array}{ccc} a_{\mathscr{B}!}y^*\mathbb{T}_{\widetilde{\mathfrak{g}}}\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[d+N] & \xrightarrow{\quad(2.15)\quad} & a_{\mathscr{B}!}t_!\underline{\Bbbk}_{\widetilde{\mathfrak{g}}}[2d] \\ \downarrow{\scriptstyle(2.11)} & & \downarrow{\scriptstyle(2.12)} \\ a_{\mathscr{B}!}y^*y_!\underline{\Bbbk}_{\mathscr{B}}[2N] & \longrightarrow & a_{\mathscr{B}!}\underline{\Bbbk}_{\mathscr{B}}[2N] \end{array}$$

where the lower horizontal map is the obvious one. Details are left to the reader. □

Theorem 4.8. *Assume that $\Bbbk$ embeds either in a $\mathbb{Q}$-algebra, or in an $\mathbb{F}_p$-algebra where p does not divide $|\mathbf{W}|$. Then:*

(1) *the map $\sigma : \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk)) \to \mathrm{End}(H^\bullet(\mathscr{B};\Bbbk))$ defined in* (4.5) *is injective;*
(2) *we have $\dot\rho^{\Bbbk} = \dot\varphi^{\Bbbk} \circ \mathsf{sgn}$.*

Proof. By Lemmas 4.5 and 4.6 we have $\gamma_c \circ \dot r_{\Bbbk} = \gamma \circ \dot r_{\Bbbk} \circ \mathsf{sgn}$. Then by Lemma 4.7 we deduce that $\sigma \circ \tilde\varphi(\Bbbk) \circ \dot r_{\Bbbk} = \sigma \circ \tilde\rho(\Bbbk) \circ \dot r_{\Bbbk} \circ \mathsf{sgn}$. Moreover, by Lemma 4.6 again and Lemma 4.3(2), these morphisms are injective. As $\tilde\varphi(\Bbbk) \circ \dot r_{\Bbbk}$ is an isomorphism, we deduce (1), and also that $\tilde\varphi(\Bbbk) \circ \dot r_{\Bbbk} = \tilde\rho(\Bbbk) \circ \dot r_{\Bbbk} \circ \mathsf{sgn}$. This proves (2), as by definition we have $\dot\rho^{\Bbbk} = \tilde\rho(\Bbbk) \circ \dot r_{\Bbbk}$ and $\dot\varphi^{\Bbbk} = \tilde\varphi(\Bbbk) \circ \dot r_{\Bbbk}$. □

5. Consequences

Reverting to a more usual notion of perverse sheaves, we let $\mathsf{Perv}_G(\mathcal{N},\Bbbk)$ denote the category of G-equivariant perverse $\Bbbk$-sheaves on $\mathcal{N}$ (assumed to be constructible with respect to the stratification by G-orbits). Let $\mathfrak{N}_{\Bbbk}$ be the set of isomorphism classes of simple objects of $\mathsf{Perv}_G(\mathcal{N},\Bbbk)$. (Abusing notation slightly, we will sometimes write $\mathcal{A} \in \mathfrak{N}_{\Bbbk}$ when we mean that $\mathcal{A}$ is such a simple object.) Let $\mathsf{Rep}(\mathbf{W},\Bbbk)$ denote the category of finitely-generated $\Bbbk[\mathbf{W}]$-modules, and let $\operatorname{Irr}\Bbbk[\mathbf{W}]$ be the set of isomorphism classes of simple objects of $\mathsf{Rep}(\mathbf{W},\Bbbk)$. (Again, we will sometimes write $E \in \operatorname{Irr}\Bbbk[\mathbf{W}]$ when we mean that E is such a simple object.) Note that both categories $\mathsf{Perv}_G(\mathcal{N},\Bbbk)$ and $\mathsf{Rep}(\mathbf{W},\Bbbk)$ are noetherian, since $\Bbbk$ is noetherian, G has finitely many orbits on $\mathcal{N}$, and $\mathbf{W}$ is finite.

5.1. Springer correspondence. One of the main objects of study in [AHR] was the functor

$$\mathbb{S}_{\dot\rho} : \mathsf{Perv}_G(\mathcal{N},\Bbbk) \to \mathsf{Rep}(\mathbf{W},\Bbbk), \quad \mathcal{A} \mapsto \mathrm{Hom}_{\mathsf{Perv}_G(\mathcal{N},\Bbbk)}(\underline{\mathsf{Spr}}(\Bbbk),\mathcal{A}),$$

where the $\mathbf{W}$-action on $\mathrm{Hom}(\underline{\mathsf{Spr}}(\Bbbk),\mathcal{A})$ is defined by $w\cdot f = f \circ \dot\rho(w^{-1})$. By [AHR, §7], the functor $\mathbb{S}_{\dot\rho}$ commutes with restriction to Levi subgroups in an appropriate sense, and it is exact (i.e. $\underline{\mathsf{Spr}}(\Bbbk)$ is projective in $\mathsf{Perv}_G(\mathcal{N},\Bbbk)$). On the other hand, the functor $\mathbb{S}_{\dot\varphi}$ defined in the same way but using the map $\dot\varphi$ was studied implicitly in [Ju1] (in its étale version using Fourier–Deligne transform) and explicitly in [Ma] (in the same setting as the present paper). We conclude from Theorem 1.1 that $\mathbb{S}_{\dot\rho}$ and $\mathbb{S}_{\dot\varphi}$ differ only by a sign twist. Thus all the results proved about either of them are valid for both. From now on, we will use the notation $\mathbb{S}$ to refer to $\mathbb{S}_{\dot\varphi}$.

Proposition 5.1. *$\mathbb{S}$ has a left adjoint $T : \mathsf{Rep}(\mathbf{W}, \Bbbk) \to \mathsf{Perv}_G(\mathcal{N}, \Bbbk)$ such that the natural transformation $\mathrm{id} \to \mathbb{S} \circ T$ is an isomorphism.*

Proof. The argument is almost the same as that in [Ga, pp. 403–404]. Let $M \in \mathsf{Rep}(\mathbf{W}, \Bbbk)$. Since $\mathsf{Rep}(\mathbf{W}, \Bbbk)$ is noetherian, there exists an exact sequence

$$\Bbbk[\mathbf{W}]^{\oplus m} \xrightarrow{\alpha} \Bbbk[\mathbf{W}]^{\oplus n} \longrightarrow M \longrightarrow 0. \tag{5.1}$$

Here α can be viewed as an $n \times m$ matrix of elements of $\Bbbk[\mathbf{W}] \xrightarrow{\sim} \mathrm{End}_{\Bbbk[\mathbf{W}]}(\Bbbk[\mathbf{W}])$ (where the latter identification sends $w \in \mathbf{W}$ to right multiplication by w^{-1}). Let $\dot{\varphi}(\alpha)$ be the $n \times m$ matrix of elements of $\mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk))$ obtained by applying $\dot{\varphi}$ to each entry of α. Define $T(M) \in \mathsf{Perv}_G(\mathcal{N}, \Bbbk)$ by the exact sequence

$$\underline{\mathsf{Spr}}(\Bbbk)^{\oplus m} \xrightarrow{\dot{\varphi}(\alpha)} \underline{\mathsf{Spr}}(\Bbbk)^{\oplus n} \longrightarrow T(M) \longrightarrow 0. \tag{5.2}$$

One can easily check that $T(M)$ represents the functor $\mathrm{Hom}_{\Bbbk[\mathbf{W}]}(M, \mathbb{S}(-))$. By a general principle [Ga, p. 346, Proposition 10], there is a unique way to define T on morphisms so that it becomes a functor that is left adjoint to $\mathbb{S}$.

Since $\dot{\varphi}$ is a ring isomorphism, we have an isomorphism $\Bbbk[\mathbf{W}] \xrightarrow{\sim} \mathbb{S}(\underline{\mathsf{Spr}}(\Bbbk))$ in $\mathsf{Rep}(\mathbf{W}, \Bbbk)$ sending $y \in \mathbf{W}$ to $\dot{\varphi}(y^{-1}) \in \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk)) = \mathbb{S}(\underline{\mathsf{Spr}}(\Bbbk))$. The induced ring isomorphism $\Bbbk[\mathbf{W}] \xrightarrow{\sim} \mathrm{End}_{\Bbbk[\mathbf{W}]}(\Bbbk[\mathbf{W}]) \xrightarrow{\sim} \mathrm{End}_{\Bbbk[\mathbf{W}]}(\mathbb{S}(\underline{\mathsf{Spr}}(\Bbbk)))$ coincides with $\mathbb{S} \circ \dot{\varphi}$. Applying the exact functor $\mathbb{S}$ to (5.2) and comparing with (5.1), we conclude that the natural transformation $\mathrm{id} \to \mathbb{S} \circ T$ is an isomorphism. □

Corollary 5.2. *$\mathbb{S}$ is a quotient functor. That is, it induces an equivalence of categories between $\mathsf{Rep}(\mathbf{W}, \Bbbk)$ and the quotient of $\mathsf{Perv}_G(\mathcal{N}, \Bbbk)$ by the kernel of $\mathbb{S}$.*

Proof. This follows from Proposition 5.1 as in [Ga, p. 374, Proposition 5]. □

Corollary 5.3 (Springer correspondence over $\Bbbk$)**.** *There is a subset $\mathfrak{N}_\Bbbk^0 \subseteq \mathfrak{N}_\Bbbk$ such that $\mathbb{S}$ induces a bijection $\mathfrak{N}_\Bbbk^0 \xrightarrow{\sim} \mathrm{Irr}\,\Bbbk[\mathbf{W}]$, and $\mathbb{S}(\mathcal{A}) = 0$ for $\mathcal{A} \in \mathfrak{N}_\Bbbk \setminus \mathfrak{N}_\Bbbk^0$.*

Proof. The only part which is not immediate from Corollary 5.2 is the statement that any $E \in \mathrm{Irr}\,\Bbbk[\mathbf{W}]$ arises as $\mathbb{S}(\mathcal{A})$ for some $\mathcal{A} \in \mathfrak{N}_\Bbbk$. But any such E is a quotient of $\Bbbk[\mathbf{W}]$ in $\mathsf{Rep}(\mathbf{W}, \Bbbk)$, hence (since T is right exact) $T(E)$ is a quotient of $T(\Bbbk[\mathbf{W}]) \cong \underline{\mathsf{Spr}}(\Bbbk)$. Any simple quotient $\mathcal{A}$ of $T(E)$ is therefore also a quotient of $\underline{\mathsf{Spr}}(\Bbbk)$, hence $\mathbb{S}(\mathcal{A})$ is nonzero. Since $\mathbb{S}$ is exact and $\mathbb{S}(T(E)) \cong E$ is simple, we conclude that $\mathbb{S}(\mathcal{A}) \cong E$ as required. This shows, incidentally, that any maximal subobject of $T(E)$ belongs to the kernel of $\mathbb{S}$, so $T(E)$ has a unique maximal subobject and $\mathcal{A}$ is the head of $T(E)$. □

We can give a formula for the inverse bijection $\mathrm{Irr}\,\Bbbk[\mathbf{W}] \xrightarrow{\sim} \mathfrak{N}_\Bbbk^0$. Recall that we have a surjective group homomorphism $\pi_1(\mathfrak{g}_{\mathrm{rs}}) \to \mathbf{W}$, so every $E \in \mathrm{Irr}\,\Bbbk[\mathbf{W}]$ determines a simple $\Bbbk$-local system $\mathcal{L}_E$ on $\mathfrak{g}_{\mathrm{rs}}$. (Here we have to choose a base point in $\mathfrak{g}_{\mathrm{rs}}$, but this choice is irrelevant for the next proposition.)

Proposition 5.4. *For any $E \in \mathrm{Irr}\,\Bbbk[\mathbf{W}]$, the element of $\mathfrak{N}_\Bbbk^0$ corresponding to E is $\mathbb{T}_{\mathfrak{g}}(j_{\mathrm{rs}})_{!*}\mathcal{L}_E[d]$ (or, strictly speaking, the restriction to $\mathcal{N}$ of this simple perverse sheaf supported on $\mathcal{N}$).*

Proof. The argument is the same as in the étale setting (see [Ju1, §6.2.4]). Namely, since π_{rs} is a Galois covering map with group $\mathbf{W}$, we have an isomorphism

$$\mathrm{Hom}_{\mathsf{Sh}(\mathfrak{g}_{\mathrm{rs}}, \Bbbk)}(\pi_{\mathrm{rs}*}\underline{\Bbbk}, \mathcal{L}_E) \cong \mathrm{Hom}_{\Bbbk[\mathbf{W}]}(\Bbbk[\mathbf{W}], E) \cong E \tag{5.3}$$

of $\Bbbk[\mathbf{W}]$-modules, where the action of $w \in \mathbf{W}$ on the middle (resp. left) term is by precomposing with right multiplication by w (resp. with $\tilde{r}_{\Bbbk}(w^{-1})$). Since $(j_{\mathrm{rs}})_{!*}$ is fully faithful, this induces an isomorphism

$$\mathrm{Hom}_{\mathsf{Perv}(\mathfrak{g},\Bbbk)}(\underline{\mathsf{Groth}}(\Bbbk), (j_{\mathrm{rs}})_{!*}\mathcal{L}_E[d]) \cong E \tag{5.4}$$

of $\Bbbk[\mathbf{W}]$-modules. Applying the equivalence $\mathbb{T}_{\mathfrak{g}}$ and using Lemma 2.2, we deduce an isomorphism of $\Bbbk[\mathbf{W}]$-modules

$$\mathrm{Hom}_{\mathsf{Perv}_G(\mathcal{N},\Bbbk)}(\underline{\mathsf{Spr}}(\Bbbk), \mathbb{T}_{\mathfrak{g}}(j_{\mathrm{rs}})_{!*}\mathcal{L}_E[d]) \cong E, \tag{5.5}$$

where $\mathbf{W}$ acts on $\underline{\mathsf{Spr}}(\Bbbk)$ via $\dot{\varphi}$. The claim follows. □

Remark 5.5. Similar descriptions of the left and right adjoints of $\mathbb{S}$ in the case that $\Bbbk$ is a field are given by [Ma, Theorem 6.2 and Proposition 7.2].

Previous definitions of the Springer correspondence have required $\Bbbk$ to be a field. When $\Bbbk = \mathbb{Q}_\ell$, the above correspondence is Springer's original version [Spr], differing by a sign twist from Lusztig's reformulation [L1, BM]. This follows from Lemma 4.6 and Theorem 4.8(1). This 'ordinary' Springer correspondence has been determined explicitly, see [Ca, Section 13.3] (where Lusztig's version is used).

When $\Bbbk$ is a finite field of characteristic $\ell > 0$, there is also the 'modular' Springer correspondence of [Ju1, Theorem 6.2.8]. It is given by the same formula as Proposition 5.4, but in the étale setting with Fourier–Deligne transform. The change from Fourier–Deligne transform to Fourier–Sato transform makes no difference to the combinatorics of the correspondence, with the obvious identification of the two versions of $\mathfrak{N}_{\Bbbk}$. Indeed, by [JLS, Section 3], the modular Springer correspondence is completely determined by the ordinary Springer correspondence and the decomposition matrix of $\mathbf{W}$, and the results used to prove this, such as [Ju1, Theorem 6.3.2], are equally valid for the Fourier-Sato transform. (The fact that the decomposition matrix is unknown in general is irrelevant for this uniqueness argument.)

The modular Springer correspondence has been explicitly determined in [Ju1] for types A_n and G_2, in [JLS] for classical types when $\ell \neq 2$, and in as-yet unpublished tables for the remaining exceptional types.

5.2. Zero weight spaces of small representations. Assume now that $\Bbbk$ is a field and that G is simple and simply-connected. Let $\check{G}$ denote the split connected reductive group over $\Bbbk$ whose root datum is dual to that of G (so that $\check{G}$ is simple of adjoint type). For any dominant coweight λ of G, there is an irreducible representation $L(\lambda)$ of $\check{G}$ with highest weight λ. The zero weight space $L(\lambda)_0$ carries a representation of the Weyl group of $\check{G}$, which can be identified with $\mathbf{W}$.

The main result of [AHR] determined the representation of $\mathbf{W}$ on $L(\lambda)_0$ in the case where λ is *small* (meaning that the convex hull of $\mathbf{W}\lambda$ does not contain $2\check{\alpha}$ for any root α). Namely, we have the following isomorphism, obtained from [AHR, (1.3)] by tensoring both sides by the sign character (recall that in the present paper, $\mathbb{S}$ means $\mathbb{S}_{\dot{\varphi}}$ whereas in [AHR] it means $\mathbb{S}_{\dot{\rho}}$):

$$L(\lambda)_0 \cong \mathbb{S}(\pi_*(\mathrm{IC}(\mathsf{Gr}^\lambda, \Bbbk)|_{\mathcal{M}})). \tag{5.6}$$

Here Gr^λ is the orbit in the affine Grassmannian of G labelled by λ, $\mathcal{M}$ is the G-stable locally closed subvariety of the affine Grassmannian defined in [AH] (which intersects $\overline{\mathsf{Gr}^\lambda}$ in an open dense subvariety for all small λ), and $\pi : \mathcal{M} \to \mathcal{N}$ is the G-equivariant finite map defined and described in [AH].

We can now express the right-hand side of (5.6) in terms of the Springer correspondence of Corollary 5.3. The nilpotent orbits contained in $\pi(\mathsf{Gr}^\lambda \cap \mathcal{M})$ for each small λ are described explicitly in [AH, Table 1, Table 6], and we have the following dichotomy [AH, Proposition 3.2]:

Proposition 5.6. *For any small coweight λ of G, one of the following occurs.*

(1) *$\pi(\mathsf{Gr}^\lambda \cap \mathcal{M})$ is a single G-orbit $\mathcal{O}_\lambda$, and π restricts to an isomorphism $\mathsf{Gr}^\lambda \cap \mathcal{M} \xrightarrow{\sim} \mathcal{O}_\lambda$.*
(2) *$\pi(\mathsf{Gr}^\lambda \cap \mathcal{M})$ is the union of two G-orbits, of which one, say $\mathcal{O}_\lambda$, contains the other in its closure, and π restricts to a non-trivial $2:1$ Galois covering $U_\lambda \to \mathcal{O}_\lambda$, where U_λ is open in $\mathsf{Gr}^\lambda \cap \mathcal{M}$.*

Corollary 5.7. *In the two cases of Proposition* 5.6*, we have, respectively:*

(1) *$L(\lambda)_0 \cong \mathbb{S}(\mathrm{IC}(\mathcal{O}_\lambda, \Bbbk))$. Hence $L(\lambda)_0$ is either simple or zero.*
(2) *Assuming that the characteristic of $\Bbbk$ is not 2, $L(\lambda)_0 \cong \mathbb{S}(\mathrm{IC}(\mathcal{O}_\lambda, \Bbbk)) \oplus \mathbb{S}(\mathrm{IC}(\mathcal{O}_\lambda, \mathcal{L}))$ where $\mathcal{L}$ denotes the nontrivial rank-one G-equivariant $\Bbbk$-local system on $\mathcal{O}_\lambda$ arising from the Galois covering $U_\lambda \to \mathcal{O}_\lambda$. Hence $L(\lambda)_0$ is either the direct sum of two nonisomorphic simples, simple, or zero.*

Proof. In the first case, the finiteness of π and Proposition 5.6(1) imply that

$$\pi_*(\mathrm{IC}(\mathsf{Gr}^\lambda, \Bbbk)|_{\mathcal{M}}) \cong \mathrm{IC}(\mathcal{O}_\lambda, \Bbbk).$$

In the second case, the finiteness of π, the assumption that the characteristic of $\Bbbk$ is not 2, and Proposition 5.6(2) imply that

$$\pi_*(\mathrm{IC}(\mathsf{Gr}^\lambda, \Bbbk)|_{\mathcal{M}}) \cong \mathrm{IC}(\mathcal{O}_\lambda, \Bbbk) \oplus \mathrm{IC}(\mathcal{O}_\lambda, \mathcal{L}).$$

The result now follows from (5.6) and Corollary 5.3. □

Remark 5.8. In the situation of Corollary 5.7(2), the G-equivariant fundamental group of $\mathcal{O}_\lambda$ is either $\mathbb{Z}/2\mathbb{Z}$ or S_3 (the latter occurring only in the case of Example 5.11 below). Hence $\mathcal{L}$ is in fact the unique nontrivial rank-one G-equivariant local system on $\mathcal{O}_\lambda$.

If $\mathrm{char}(\Bbbk) = 0$, Corollary 5.7 recovers the calculations in [R1, R2], as noted in [AHR]. Now we consider the case where $\mathrm{char}(\Bbbk) = \ell > 0$.

Example 5.9. If $G = SL_n$, we are always in case (1). Let $\lambda = (a_1, \dots, a_n)$ where $a_1 \geq \dots \geq a_n \geq -1$ and $\sum_{i=1}^n a_i = 0$. Let $\hat{\lambda} = (a_1+1, \dots, a_n+1)$ be the associated partition of n. Then the orbit denoted $\mathcal{O}_\lambda$ above consists of nilpotent matrices with Jordan type $\hat{\lambda}$. Using the determination of the modular Springer correspondence in this case [Ju1, Theorem 6.4.1], we recover the well-known fact [Ja1, A.23(5)] that $L(\lambda)_0 \simeq \mathbf{D}^{\hat{\lambda}^t}$ when λ is ℓ-restricted, and 0 otherwise. (We use the standard notation for the simple modules of $\Bbbk[S_n]$.)

Example 5.10. Suppose G is of type B, C or D and $\ell > 2$. The modular Springer correspondence for this case has been determined in [JLS, Theorem 4.21]. Combining that result with [AH, Table 1], we obtain the explicit description of zero weight spaces of small representations displayed in Table 1. We use the notation of [AH] for the small coweights, and the notation of [JLS] for the simple $\Bbbk[\mathbf{W}]$-modules. If $\ell = 3$ and G is of type B_3, one should interpret $\mathbf{D}^{((1^3),\emptyset)}$ as 0. Among all the cases considered in the present example, this is the only time that one of the intersection cohomology complexes appearing in Corollary 5.7 is killed by $\mathbb{S}$.

G	λ		$L(\lambda)_0$
B_n	$(21^{2j}0^{n-2j-1})$	(for $1 \le j \le \frac{n-1}{2}$)	$\mathbf{D}^{((n-j-1,j),(1))} \oplus \mathbf{D}^{((n-j-1,j,1),\emptyset)}$
	(20^{n-1})		$\mathbf{D}^{((n-1),(1))}$
	$(1^{2j}0^{2n-j})$		$\mathbf{D}^{((n-j,j),\emptyset)}$
C_n	(1^j0^{n-j})		$\mathbf{D}^{((n-\frac{j}{2}),(\frac{j}{2}))}$ if j even $\mathbf{D}^{((\frac{j-1}{2}),(n-\frac{j-1}{2}))}$ if j odd
D_n	$(21^{n-2}(\pm 1))$	(if n odd)	$\mathbf{E}^{[(\frac{n-1}{2},1),(\frac{n-1}{2})]}$
	$(21^{2j}0^{n-2j-1})$	(for $1 \le j < \frac{n-1}{2}$)	$\mathbf{E}^{[(n-j-1,1),(j)]} \oplus \mathbf{E}^{[(n-j-1),(j,1)]}$
	(20^{n-1})		$\mathbf{E}^{[(n-1,1),\emptyset]}$
	$(1^{n-1}(\pm 1))$	(if n even)	$\mathbf{E}^{[(\frac{n}{2}),\pm]}$
	$(1^{2j}0^{n-2j})$	(for $0 \le j < \frac{n}{2}$)	$\mathbf{E}^{[(n-j),(j)]}$

TABLE 1. Zero weight spaces of small representations for classical groups

Example 5.11. Assume G is of type G_2 and $\ell > 2$. The modular Springer correspondence for this type is described in [Ju1, §7.7]. Let λ be the higher fundamental coweight, so that $L(\lambda)$ is either the adjoint representation of $\check{G}$ or its unique simple quotient if $\ell = 3$. We are in case (2), $\mathcal{O}_\lambda$ is the subregular orbit, $\mathbb{S}(\mathrm{IC}(\mathcal{O}_\lambda, \mathcal{L})) = 0$, and accordingly $L(\lambda)_0 \cong \mathbb{S}(\mathrm{IC}(\mathcal{O}_\lambda, \Bbbk))$ (of dimension 1 if $\ell = 3$ and 2 if $\ell > 3$).

Example 5.12. Among the cases where G is of exceptional type and $\ell > 2$, there are only two occasions other than that described in the preceding example where $\mathbb{S}$ kills one of the intersection cohomology complexes appearing in Corollary 5.7. Namely, if $\lambda = 3\check{\omega}_1$ or $3\check{\omega}_6$ in type E_6 (in the numbering of [Bou]), Corollary 5.7(1) says that $L(\lambda)_0 \simeq \mathbb{S}(\mathrm{IC}(2A_2, \Bbbk))$. If $\ell = 3$ the latter vanishes; in this case $L(\lambda)$ is the Frobenius twist of a minuscule representation.

Since there are only finitely many small coweights in the exceptional types, one can compute in every case the image of $L(\lambda)_0$ in the Grothendieck group of $\mathsf{Rep}(\mathbf{W}, \Bbbk)$, using the known answer in characteristic zero [R1, R2] together with the known decomposition matrices for the small representations of $\check{G}$ [Lü] and the decomposition matrices for the exceptional Weyl groups which are available in GAP3. (The latter decomposition matrices were first computed in [Kh] for $\ell = 3$ and [KhM] for $\ell > 3$.) As predicted by Corollary 5.7, when $\ell > 2$ one finds that $L(\lambda)_0$ is multiplicity-free and has at most two simple constituents. Note that in the case where two simple constituents appear, Corollary 5.7(2) gives the additional information that $L(\lambda)_0$ is the direct sum of the two.

If we are in the situation of Proposition 5.6(2) and $\ell = 2$, it is possible for $L(\lambda)_0$ to have length greater than 2. (For example, in type E_7 the $\Bbbk[\mathbf{W}]$-module $L(\check{\omega}_3)_0$ has four distinct simple constituents, three of which appear with multiplicity 2, and the last one with multiplicity 1.) In geometric terms, the reason that characteristic 2 is different is that the sheaf $(\pi|_{U_\lambda})_* \underline{\Bbbk}_{U_\lambda}$ is a non-split extension of $\underline{\Bbbk}_{\mathcal{O}_\lambda}$ by itself. Since the intermediate extension functor preserves socles and heads [Ju2, Proposition 2.28], one can conclude that $\pi_*(\mathrm{IC}(\mathsf{Gr}^\lambda, \Bbbk)|_{\mathcal{M}})$ has socle and head equal to $\mathrm{IC}(\mathcal{O}_\lambda, \Bbbk)$. However, it may also have additional simple constituents in the middle, supported on $\overline{\mathcal{O}_\lambda} \setminus \mathcal{O}_\lambda$. When one then applies the functor $\mathbb{S}$ to obtain $L(\lambda)_0$, diverse situations occur: there are cases where the head and socle are killed by $\mathbb{S}$ and the constituents in the middle remain, and others where only the head and socle survive.

6. Étale version

The arguments of this paper can be adapted to prove an analogue of the main result in the étale topology over a finite field, substituting the Fourier–Deligne transform for the Fourier–Sato transform. In fact, the 'change of rings' operation is simpler (since it is at the heart of the construction of the derived categories); in particular we don't need the functorial point of view. In this section, we describe the set-up, and we briefly indicate some of the points that require special attention.

6.1. Derived categories of sheaves. Fix a finite field $\mathbb{F}_q$ of characteristic p, and a prime number $\ell \neq p$. Let X be a variety over $\mathbb{F}_q$.

We say that a ring $\Bbbk$ is *admissible* if it is isomorphic to a finite integral extension of some $\mathbb{Z}/\ell^n\mathbb{Z}$ or of $\mathbb{Z}_\ell$, or else to a (possibly infinite) integral extension of $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$. For each of these rings, there is a triangulated category $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$, to be thought of as the category of 'constructible complexes of $\Bbbk$-sheaves'. The definition of $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ is rather complicated when $\Bbbk$ is not a finite ring; see [KW, Appendix A] for an exposition. One special case will be recalled in Section 6.3. In each case, $D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk)$ has a 'natural' t-structure and a perverse t-structure, with hearts $\mathsf{Sh}_{\mathrm{c}}(X, \Bbbk)$ and $\mathsf{Perv}(X, \Bbbk)$, respectively. We have a constant sheaf $\underline{\Bbbk}_X \in \mathsf{Sh}_{\mathrm{c}}(X, \Bbbk)$.

Suppose now that $\psi : \mathbb{F}_q \to \Bbbk^\times$ is a nontrivial homomorphism. If E and E^* are dual vector bundles over an $\mathbb{F}_q$-variety Y, there is a *Fourier–Deligne transform*, denoted $\mathbb{T}^\psi_E : D^{\mathrm{b}}_{\mathrm{c}}(E, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{c}}(E^*, \Bbbk)$. Its theory is developed in the generality we need in [Ju1, Chap. 5].

6.2. Springer and Grothendieck functors. Let G be a split connected reductive group over $\mathbb{F}_q$. The varieties $\mathscr{B}$, $\mathcal{N}$, $\widetilde{\mathcal{N}}$, etc., are then also defined over $\mathbb{F}_q$, and we can define objects

$$\underline{\mathsf{Groth}}(\Bbbk) \in \mathsf{Perv}(\mathfrak{g}, \Bbbk), \qquad \underline{\mathsf{Spr}}(\Bbbk) \in \mathsf{Perv}(\mathcal{N}, \Bbbk).$$

We also assume that $\mathfrak{g}$ can be equipped with a G-invariant nondegenerate bilinear form, and we fix such a form. (This assumption imposes a constraint on p; see [Le, §2.5].) Having identified $\mathfrak{g} \cong \mathfrak{g}^*$, we regard $\mathbb{T}^\psi_{\mathfrak{g}}$ as an autoequivalence

$$\mathbb{T}^\psi_{\mathfrak{g}} : D^{\mathrm{b}}_{\mathrm{c}}(\mathfrak{g}, \Bbbk) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{c}}(\mathfrak{g}, \Bbbk).$$

From [Ju1, §6.2], we have étale analogues of Lemmas 2.1 and 2.2, although the latter now includes a Tate twist:

$$i^*_{\mathcal{N}} \underline{\mathsf{Groth}}(\Bbbk)[-r] \cong \underline{\mathsf{Spr}}(\Bbbk), \qquad \mathbb{T}^\psi_{\mathfrak{g}} \circ \underline{\mathsf{Groth}}(\Bbbk) \cong i_{\mathcal{N}!} \circ \underline{\mathsf{Spr}}(\Bbbk)(-N - r).$$

One can then define analogues of $\dot{\rho}^{\Bbbk}$ and $\dot{\varphi}^{\Bbbk}$, see [Ju1, Theorem 6.2.1, Corollary 6.2.5].

6.3. Change of rings. Let us assume that $\Bbbk$ is a finite integral extension of $\mathbb{Z}_\ell$, with maximal ideal $\mathfrak{m}$. Any quotient $\Bbbk/\mathfrak{m}^n$ is a finite ring. For $n \geq 1$, we consider the full triangulated subcategory $D^{\mathrm{b}}_{\mathrm{ctf}}(X, \Bbbk/\mathfrak{m}^n) \subset D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk/\mathfrak{m}^n)$ whose objects are quasiisomorphic to a bounded complex of flat sheaves of $\Bbbk/\mathfrak{m}^n$-modules. Then

$$D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk) = \varprojlim_n D^{\mathrm{b}}_{\mathrm{ctf}}(X, \Bbbk/\mathfrak{m}^n) \tag{6.1}$$

by definition. If $\Bbbk'$ is any quotient of $\Bbbk$, we define the functor

$$\mathrm{E}^X_{\Bbbk,\Bbbk'} : D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \Bbbk')$$

to be the composition of the canonical functor $D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk) \to D^{\mathrm{b}}_{\mathrm{ctf}}(X,\Bbbk')$ with the inclusion $D^{\mathrm{b}}_{\mathrm{ctf}}(X,\Bbbk') \to D^{\mathrm{b}}_{\mathrm{c}}(X,\Bbbk')$. This functor is an 'extension of scalars' functor rather than a forgetful functor as in §2.3. Sheaf functors $(\cdot)_!$, $(\cdot)^*$ and the tensor product on $D^{\mathrm{b}}_{\mathrm{c}}(\cdot,\Bbbk)$, as well as the usual isomorphisms between them (composition, base-change, projection formula) are defined termwise with respect to the inverse limit (6.1), so that they are compatible with the functors $\mathrm{E}_{\Bbbk,\Bbbk'}$ in the natural sense. Similar remarks apply to the Fourier–Deligne transform (which is defined in terms of the preceding operations) and its compatibility properties similar to (2.14), (2.13). Moreover, one clearly has $\mathrm{E}^X_{\Bbbk,\Bbbk'}(\underline{\Bbbk}_X) = \underline{\Bbbk'}_X$.

The following result follows from these remarks:

Proposition 6.1. *Suppose that $\Bbbk$ is a finite integral extension of $\mathbb{Z}_\ell$, and that $\Bbbk'$ is a quotient of $\Bbbk$. Then there is a canonical $\Bbbk$-algebra homomorphism*

$$\mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk)) \to \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk'))$$

induced by the functor $\mathrm{E}^{\mathcal{N}}_{\Bbbk,\Bbbk'}$. Moreover, the following diagrams commute:

$$\begin{array}{ccc} \Bbbk[\mathbf{W}] & \xrightarrow{\dot\rho^{\Bbbk}} & \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk)) \\ \downarrow & & \downarrow \\ \Bbbk'[\mathbf{W}] & \xrightarrow{\dot\rho^{\Bbbk'}} & \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk')) \end{array} \qquad \begin{array}{ccc} \Bbbk[\mathbf{W}] & \xrightarrow{\dot\varphi^{\Bbbk}} & \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk)) \\ \downarrow & & \downarrow \\ \Bbbk'[\mathbf{W}] & \xrightarrow{\dot\varphi^{\Bbbk'}} & \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk')) \end{array}$$

6.4. Main result. The following statement is analogous to Theorem 3.7.

Theorem 6.2. *Let $\Bbbk$ be an admissible ring. The two maps $\dot\rho^{\Bbbk}, \dot\varphi^{\Bbbk} : \Bbbk[\mathbf{W}] \to \mathrm{End}(\underline{\mathsf{Spr}}(\Bbbk))$ are related as follows: $\dot\rho^{\Bbbk} = \dot\varphi^{\Bbbk} \circ \mathsf{sgn}$.*

Proof. In view of Proposition 6.1, it suffices to treat the cases where $\Bbbk$ is either an integral extension of $\mathbb{Z}_\ell$ or an algebraic extension of $\mathbb{Q}_\ell$. Those cases are covered by the classical arguments explained in Section 4. □

Appendix A. Isomorphisms of sheaf functors

In this appendix, we collect a few lemmas on compatibility of various isomorphisms of sheaf functors with change of rings, and compatibility between isomorphisms arising from Fourier–Sato transform. Throughout, $\Bbbk'$ denotes a $\Bbbk$-algebra.

A.1. Composition and change of rings. Suppose we have two maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{A.1}$$

of topological spaces, and let $h = g \circ f$.

Lemma A.1. *In the setting of* (A.1)*, the following diagrams commute:*

$$\begin{array}{ccc} \mathrm{For}^Z_{\Bbbk,\Bbbk'} g_! f_! & \xrightarrow{(2.2)} & g_! \mathrm{For}^Y_{\Bbbk,\Bbbk'} f_! \\ \downarrow{\scriptstyle (\mathrm{Co})} & & \downarrow{\scriptstyle (2.2)} \\ & & g_! f_! \mathrm{For}^X_{\Bbbk,\Bbbk'} \\ & & \downarrow{\scriptstyle (\mathrm{Co})} \\ \mathrm{For}^Z_{\Bbbk,\Bbbk'} h_! & \xrightarrow[(2.2)]{} & h_! \mathrm{For}^X_{\Bbbk,\Bbbk'} \end{array} \qquad \begin{array}{ccc} \mathrm{For}^X_{\Bbbk,\Bbbk'} f^* g^* & \xrightarrow{(2.3)} & f^* \mathrm{For}^Y_{\Bbbk,\Bbbk'} g^* \\ \downarrow{\scriptstyle (\mathrm{Co})} & & \downarrow{\scriptstyle (2.3)} \\ & & f^* g^* \mathrm{For}^Z_{\Bbbk,\Bbbk'} \\ & & \downarrow{\scriptstyle (\mathrm{Co})} \\ \mathrm{For}^X_{\Bbbk,\Bbbk'} h^* & \xrightarrow[(2.3)]{} & h^* \mathrm{For}^Z_{\Bbbk,\Bbbk'} \end{array}$$

Proof. By general principles, it suffices to check the corresponding statements for nonderived functors of abelian categories, and those are straightforward. □

A.2. Base change and change of rings. Consider a cartesian square of topological spaces:

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array} \tag{A.2}$$

Lemma A.2. *In the setting of* (A.2)*, the following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{For}^Y_{\Bbbk,\Bbbk'} g^* f_! & \xrightarrow{(2.3)} & g^* \mathrm{For}^Z_{\Bbbk,\Bbbk'} f_! & \xrightarrow{(2.2)} & g^* f_! \mathrm{For}^X_{\Bbbk,\Bbbk'} \\ {\scriptstyle (\mathrm{BC})}\downarrow & & & & \downarrow{\scriptstyle (\mathrm{BC})} \\ \mathrm{For}^Y_{\Bbbk,\Bbbk'} f'_! g'^* & \xrightarrow{(2.2)} & f'_! \mathrm{For}^W_{\Bbbk,\Bbbk'} g'^* & \xrightarrow{(2.3)} & f'_! g'^* \mathrm{For}^X_{\Bbbk,\Bbbk'} \end{array}$$

Proof. As above, it suffices to prove the analogous nonderived statement. The reader is referred to [KaS, Proposition 2.5.11] for an explicit description of the nonderived base-change isomorphism. It is easily seen that the construction there commutes with forgetful functors in the desired way. □

A.3. Constant sheaf and change of rings. Let $f : X \to Y$ be a continuous map of topological spaces. The following lemma is obvious.

Lemma A.3. *The following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{For}^X_{\Bbbk,\Bbbk'} f^* \underline{(-)}_Y & \xrightarrow{(2.3)} & f^* \mathrm{For}^Y_{\Bbbk,\Bbbk'} \underline{(-)}_Y & \xrightarrow{(2.4)} & f^* \underline{(-)}_Y \mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'} \\ {\scriptstyle (\mathrm{Const})}\downarrow & & & & \downarrow{\scriptstyle (\mathrm{Const})} \\ \mathrm{For}^X_{\Bbbk,\Bbbk'} \underline{(-)}_X & & \xrightarrow{\qquad (2.4) \qquad} & & \underline{(-)}_X \mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'} \end{array}$$

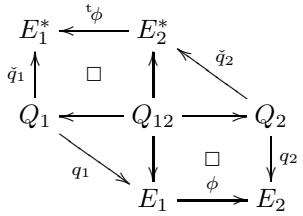
A.4. Fourier transform and change of rings. Consider the context of (2.13). This isomorphism is deduced from base-change and composition isomorphisms using the following diagram where Q' is to E' what Q is to E:

$$\begin{array}{ccccc} E' & \xleftarrow{q'} & Q' & \xrightarrow{\check{q}'} & (E')^* \\ {\scriptstyle f}\downarrow & \square & \downarrow & \square & \downarrow{\scriptstyle f'} \\ E & \xleftarrow[q]{} & Q & \xrightarrow[\check{q}]{} & E^* \end{array}$$

Lemma A.4. *In the context of* (2.13)*, the following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{For}^{E^*}_{\Bbbk,\Bbbk'} \mathbb{T}_E f_! & \xrightarrow{(2.10)} & \mathbb{T}_E \mathrm{For}^E_{\Bbbk,\Bbbk'} f_! & \xrightarrow{(2.2)} & \mathbb{T}_E f_! \mathrm{For}^E_{\Bbbk,\Bbbk'} \\ {\scriptstyle (2.13)}\downarrow & & & & \downarrow{\scriptstyle (2.13)} \\ \mathrm{For}^{E^*}_{\Bbbk,\Bbbk'} f'_! \mathbb{T}_{E'} & \xrightarrow{(2.2)} & f'_! \mathrm{For}^{E^*}_{\Bbbk,\Bbbk'} \mathbb{T}_{E'} & \xrightarrow{(2.10)} & f'_! \mathbb{T}_{E'} \mathrm{For}^E_{\Bbbk,\Bbbk'} \end{array}$$

Now, consider the context of (2.14). This isomorphism is deduced from base-change and composition isomorphisms using the following diagram, where both squares are cartesian and both triangles are commutative:



Lemma A.5. *In the context of* (2.14)*, the following diagram commutes:*

$$\begin{array}{ccccc}
\mathrm{For}_{\Bbbk,\Bbbk'}^{E_2^*}\,{}^{\mathrm{t}}\phi^*\mathbb{T}_{E_1}[-\operatorname{rk}E_1] & \xrightarrow{(2.3)} & {}^{\mathrm{t}}\phi^*\mathrm{For}_{\Bbbk,\Bbbk'}^{E_1^*}\mathbb{T}_{E_1}[-\operatorname{rk}E_1] & \xrightarrow{(2.10)} & {}^{\mathrm{t}}\phi^*\mathbb{T}_{E_1}\mathrm{For}_{\Bbbk,\Bbbk'}^{E_1}[-\operatorname{rk}E_1] \\
\downarrow{\scriptstyle (2.14)} & & & & \downarrow{\scriptstyle (2.14)} \\
\mathrm{For}_{\Bbbk,\Bbbk}^{E_2^*}\mathbb{T}_{E_2}\phi_!\,[-\operatorname{rk}E_2] & \xrightarrow{(2.10)} & \mathbb{T}_{E_2}\mathrm{For}_{\Bbbk,\Bbbk'}^{E_2}\phi_!\,[-\operatorname{rk}E_2] & \xrightarrow{(2.2)} & \mathbb{T}_{E_2}\phi_!\mathrm{For}_{\Bbbk,\Bbbk'}^{E_1}[-\operatorname{rk}E_2]
\end{array}$$

Proofs of Lemmas A.4 and A.5. The claims follow from Lemmas A.1 and A.2. □

A.5. Fourier transform of the constant sheaf. Consider the context of (2.11). By definition, we have $\mathbb{T}_E(\underline{M}_E) = \check{q}_!q^*\underline{M}_E[\operatorname{rk}E] = \check{q}_!\underline{M}_Q[\operatorname{rk}E]$. Let $U = \check{q}^{-1}(E^* \smallsetminus Y)$, and let $h : U \to Q$ and $i : Q \smallsetminus U \to Q$ be the inclusion maps. It is easy to see that the complement $Q \smallsetminus U$ can be identified with E, and $\check{q} \circ i$ can be identified with $y \circ p : E \to Y \to E^*$. Form the natural distinguished triangle

$$h_!\underline{M}_U \to \underline{M}_Q \to i_!\underline{M}_E \to .$$

It is easily checked that $\check{q}_!h_!(\underline{M}_U) = 0$. Thus, the adjunction morphism

$$\underline{M}_Q \to i_!i^*\underline{M}_Q = i_!\underline{M}_E \tag{A.3}$$

becomes an isomorphism after applying the functor $\check{q}_!$. Next, we have a pair of composition isomorphisms

$$\check{q}_!i_!\underline{M}_E \xrightarrow{\sim} (\check{q} \circ i)_!\underline{M}_E \xrightarrow{\sim} y_!p_!\underline{M}_E. \tag{A.4}$$

Finally, as explained in Lemma A.7 below there is a canonical isomorphism

$$p_!(\underline{M}_E) \cong \underline{M}_Y[-2\operatorname{rk}E]. \tag{A.5}$$

The isomorphism (2.11) is obtained by composing (A.3), (A.4), and (A.5).

Lemma A.6. *The following diagram commutes:*

$$\begin{array}{ccccc}
\mathrm{For}_{\Bbbk,\Bbbk'}^{E^*}\mathbb{T}_E\underline{(-)}_E & \xrightarrow{(2.10)} & \mathbb{T}_E\mathrm{For}_{\Bbbk,\Bbbk'}^{E}\underline{(-)}_E & \xrightarrow{(2.4)} & \mathbb{T}_E\underline{(-)}_E\mathrm{For}_{\Bbbk,\Bbbk'}^{\mathrm{m}} \\
\downarrow{\scriptstyle (2.11)} & & & & \downarrow{\scriptstyle (2.11)} \\
\mathrm{For}_{\Bbbk,\Bbbk'}^{E^*}y_!\underline{(-)}_Y[-\operatorname{rk}E] & \xrightarrow{(2.2)} & y_!\mathrm{For}_{\Bbbk,\Bbbk'}^{Y}\underline{(-)}_Y[-\operatorname{rk}E] & \xrightarrow{(2.4)} & y_!\underline{(-)}_Y\mathrm{For}_{\Bbbk,\Bbbk'}^{\mathrm{m}}[-\operatorname{rk}E]
\end{array}$$

Proof. To show that (2.11) commutes with $\mathrm{For}_{\Bbbk,\Bbbk'}$, we must show the same statement for each of (A.3), (A.4), and (A.5). For (A.4), this is Lemma A.1. For (A.3),

this follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{For}^Q_{\Bbbk,\Bbbk'} & \longrightarrow & \mathrm{For}^Q_{\Bbbk,\Bbbk'} i_! i^* \\ \downarrow & & \downarrow{\scriptstyle (2.2)} \\ i_! i^* \mathrm{For}^Q_{\Bbbk,\Bbbk'} & \xrightarrow[(2.3)]{} & i_! \mathrm{For}^E_{\Bbbk,\Bbbk'} i^* \end{array}$$

which can be checked at the level of abelian categories, as in Lemmas A.1 and A.2. Finally, for (A.5) this is proved in Lemma A.7 below. □

A.6. Cohomology with compact supports and change of rings. Consider the setting of §A.5.

Lemma A.7. *There exists an isomorphism of functors* $p_!\underline{(-)}_E \cong \underline{(-)}_Y[-2\,\mathrm{rk}\,E]$, *see* (2.12). *Moreover, the following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{For}^Y_{\Bbbk,\Bbbk'} p_! \underline{(-)}_E & \xrightarrow{(2.2)} & p_! \mathrm{For}^E_{\Bbbk,\Bbbk'} \underline{(-)}_E & \xrightarrow{(2.4)} & p_! \underline{(-)}_E \mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'} \\ \downarrow{\scriptstyle (2.12)} & & & & \downarrow{\scriptstyle (2.12)} \\ \mathrm{For}^Y_{\Bbbk,\Bbbk'} \underline{(-)}_Y[-2\,\mathrm{rk}\,E] & & \xrightarrow{(2.4)} & & \underline{(-)}_Y \mathrm{For}^{\mathrm{m}}_{\Bbbk,\Bbbk'}[-2\,\mathrm{rk}\,E] \end{array}$$

Proof. First, assume that the vector bundle is trivial: $E = \mathbb{C}^{\mathrm{rk}\,E} \times Y$. Then the arguments in the proof of [KaS, Proposition 3.2.3(iii)] can be easily adapted to produce our isomorphism (2.12), and the commutativity of the diagram is obvious. If now ϕ is an automorphism of E (as a complex vector bundle over Y) then ϕ induces an automorphism of the functor $p_!\underline{(-)}_E$. However this automorphism is trivial by the arguments of the proof of [KaS, Proposition 3.2.3(iv)].

Now assume that E can be trivialized. Then by the claim about automorphisms above, there exists a canonical isomorphism (2.12) (i.e. any choice of a trivialization produces such an isomorphism, and it does not depend on the trivialization). The commutativity of the diagram also follows from the case E is trivial.

Finally, consider the general case. Let M be in $\Bbbk$-mod. Then the object $p_!\underline{M}_E[2\,\mathrm{rk}\,E]$ of $D^{\mathrm{b}}(Y,\Bbbk)$ restricts to a sheaf on any open subset Y' of Y over which E is trivializable. It follows that this object itself is a sheaf. Moreover, the restriction of this sheaf to Y' is canonically isomorphic to $\underline{M}_{Y'}$ for any Y' as above. The canonicity implies that these isomorphisms can be glued to an isomorphism $p_!\underline{M}_E[2\,\mathrm{rk}\,E] \cong \underline{M}_Y$, which is clearly functorial in M. Finally, the commutativity of the diagram follows from the commutativity on any open subset on which E is trivializable, which was checked above. □

A.7. Compatibilities involving Fourier transform.

Lemma A.8. *In the context of* (2.11), *the following diagram commutes, where the bottom horizontal morphism is the obvious one:*

$$\begin{array}{ccc} y^* \mathbb{T}_E \underline{(-)}_E & \xrightarrow{(2.15)} & p_! \underline{(-)}_E[\mathrm{rk}\,E] \\ \downarrow{\scriptstyle (2.11)} & & \downarrow{\scriptstyle (2.12)} \\ y^* y_! \underline{(-)}_Y[-\mathrm{rk}\,E] & \longrightarrow & \underline{(-)}_Y[-\mathrm{rk}\,E] \end{array}$$

Proof. Consider the following diagram, in which the spaces and maps are defined in either §2.7 or §A.5. The commutative square on the left-hand side is cartesian.

$$\begin{array}{ccccc} & & \overset{\mathrm{id}}{\frown} & & \\ E & \xrightarrow[i]{} & Q & \xrightarrow[q]{} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \check{q}} & & \\ Y & \xrightarrow{y} & E^* & & \end{array}$$

Note that both i and y are closed inclusions, so we have adjunction morphisms $\eta : \mathrm{id} \to i_! i^*$ and $\epsilon : y^* y_! \to \mathrm{id}$. We claim that the following diagram commutes:

$$\begin{array}{ccccc} y^*\check{q}_! & \xrightarrow{\eta} & y^*\check{q}_! i_! i^* & \xrightarrow{\text{(Co)}} & y^* y_! p_! i^* \\ {\scriptstyle \text{(BC)}}\downarrow & & & \overset{\epsilon}{\swarrow} & \\ p_! i^* & & & & \end{array} \tag{A.6}$$

It suffices to check this at the level of abelian categories, and that can be done using the explicit descriptions of the various morphisms found in [KaS, Chapter 2].

Next, unravelling the definitions of (2.11) and (2.15), we find that we must prove the commutativity of the following diagram. Here, for brevity, we put $r = -2 \operatorname{rk} E$.

$$\begin{array}{ccccccc} & & \overset{(2.11)}{\frown} & & & & \overset{(2.12)}{\frown} \\ y^*\check{q}_! q^* \underline{(-)}_E & \xrightarrow{\eta} & y^*\check{q}_! i_! i^* q^* \underline{(-)}_E & \xrightarrow{\text{(Co)}} & y^* y_! p_! i^* q^* \underline{(-)}_E & \xrightarrow{\text{(Co)}} & y^* y_! p_! \underline{(-)}_E \xrightarrow{(2.12)} y^* y_! \underline{(-)}_Y[r] \\ {\scriptstyle \text{(BC)}}\downarrow & \overset{(2.15)}{\searrow} & & \overset{\epsilon}{\swarrow} & & & \downarrow\epsilon \qquad \downarrow\epsilon \\ p_! i^* q^* \underline{(-)}_E & & \xrightarrow[\text{(Co)}]{} & & & & p_! \underline{(-)}_E \xrightarrow[(2.12)]{} \underline{(-)}_Y[r] \end{array}$$

The commutativity of the upper left-hand part follows from (A.6), and the commutativity of the portion bounded by the arrows labelled 'ϵ' is obvious. □

The following two lemmas can be deduced from the basic compatibilities between (Co) and (BC) established in [AHR, Lemmas B.4, B.6, B.7 and B.8]. The proofs are routine applications of the general techniques of [AHR, Appendix B]; the details are left to the reader.

First, let $\phi : E_1 \to E_2$ be a morphism of vector bundles, and let $g : Y' \to Y$ be a continuous map. Then we can set $E_1' := Y' \times_Y E_1$, $E_2' := Y' \times_Y E_2$, and consider the natural maps

$$f_1 : E_1' \to E_1,\ f_1' : (E_1')^* \to E_1^*,\ f_2 : E_2' \to E_2,\ f_2' : (E_2')^* \to E_2^*,\ \phi' : E_1' \to E_2'.$$

We set $r_1 := \operatorname{rk} E_1$ and $r_2 := \operatorname{rk} E_2$.

Lemma A.9. *The following diagram commutes:*

$$\begin{array}{ccccc} ({}^{\mathrm{t}}\phi)^* \mathbb{T}_{E_1} f_{1!}[-r_1] & \xrightarrow{(2.14)} & \mathbb{T}_{E_2} \phi_! f_{1!}[-r_2] & \xrightarrow{\text{(Co)}} & \mathbb{T}_{E_2} f_{2!} \phi'_![-r_2] \\ {\scriptstyle (2.13)}\downarrow & & & & \downarrow{\scriptstyle (2.13)} \\ ({}^{\mathrm{t}}\phi)^* f'_{1!} \mathbb{T}_{E'_1}[-r_1] & \xrightarrow{\text{(BC)}} & f'_{2!} ({}^{\mathrm{t}}\phi')^* \mathbb{T}_{E'_1}[-r_1] & \xrightarrow{(2.14)} & f'_{2!} \mathbb{T}_{E'_2} \phi'_![-r_2] \end{array}$$

Now, consider two morphisms of vector bundles $\phi : E_1 \to E_2$ and $\psi : E_2 \to E_3$. We set $r_1 = \operatorname{rk} E_1$, $r_2 = \operatorname{rk} E_2$, and $r_3 = \operatorname{rk} E_3$.

Lemma A.10. *The following diagram commutes:*

$$\begin{array}{ccccc} {}^{\mathrm{t}}\psi^*{}^{\mathrm{t}}\phi^*\mathbb{T}_{E_1}[-r_1] & \xrightarrow{\text{(Co)}} & {}^{\mathrm{t}}(\psi\circ\phi)^*\mathbb{T}_{E_1}[-r_1] & \xrightarrow{(2.14)} & \mathbb{T}_{E_3}(\psi\circ\phi)_!\,[-r_3] \\ \downarrow{\scriptstyle (2.14)} & & & & \downarrow{\scriptstyle \text{(Co)}} \\ {}^{\mathrm{t}}\psi^*\mathbb{T}_{E_2}\phi_!\,[-r_2] & & \xrightarrow{\qquad\qquad (2.14) \qquad\qquad} & & \mathbb{T}_{E_3}\psi_!\phi_!\,[-r_3] \end{array}$$

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