

Exceptional values of p-adic analytic functions and derivatives

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Abstract Let K be an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value $|\cdot|$. Given a meromorphic function f in K (resp. inside an "open" disk D) we check that the field of small meromorphic functions in K (resp. inside D) is algebraically closed in the whole field of meromorphic functions in K (resp. inside D). If two analytic functions h, l in K , other than affine functions, satisfy $h'l - hl' = c \in K$, then $c = 0$. The space of the entire functions solutions of the equation $y'' = \phi y$, with ϕ a meromorphic function in K or an unbounded meromorphic function in D , is at most of dimension 1. If a meromorphic function in K has no multiple pole, then f' has no exceptional value. Let f be a meromorphic function having finitely many zeroes. Then for every $c \neq 0$, $f' - c$ has an infinity of zeroes. If $\frac{1}{f}$ is not a constant or an affine function and if f has no simple pole with a residue equal to 1, then $f' + f^2$ admits at least one zero. When the field K has residue characteristic zero, then we can extend to analytic functions in D some results showed for entire functions.

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Notation and definitions: Let K be an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value $|\cdot|$. We set $K^* = K \setminus \{0\}$. Given $\alpha \in K$ and $R \in \mathbb{R}_+^*$, we denote by $d(\alpha, R)$ the disk $\{x \in K \mid |x - \alpha| \leq R\}$ and by $d(\alpha, R^-)$ the disk $\{x \in K \mid |x - \alpha| < R\}$, by $\mathcal{A}(K)$ the K -algebra of analytic functions in K (i.e. the set of power series with an infinite radius of convergence) and by $\mathcal{A}(d(\alpha, R^-))$ the K -algebra of analytic functions in $d(\alpha, R^-)$ (i.e. the set of power series in $(x - \alpha)$ with a radius of convergence $r \geq R$).

We denote by $\mathcal{M}(K)$ the field of meromorphic functions in K (i.e. the field of fractions of $\mathcal{A}(K)$) and by $\mathcal{M}(d(\alpha, R^-))$ the field of meromorphic functions in $d(\alpha, R^-)$ (i.e. the field of fractions of $\mathcal{A}(d(\alpha, R^-))$).

Similarly, we denote by $\mathcal{A}_b(d(\alpha, R^-))$ the K -algebra of bounded analytic functions in $d(\alpha, R^-)$ and by $\mathcal{M}_b(d(\alpha, R^-))$ the field of fractions of $\mathcal{A}_b(d(\alpha, R^-))$. On $\mathcal{A}_b(d(\alpha, R^-))$ we denote by $\|\cdot\|_{d(\alpha, R^-)}$ the norm of uniform convergence on $d(\alpha, R^-)$.

Next we set $\mathcal{A}_u(d(\alpha, R^-)) = \mathcal{A}(d(\alpha, R^-)) \setminus \mathcal{A}_b(d(\alpha, R^-))$ and $\mathcal{M}_u(d(\alpha, R^-)) = \mathcal{M}(d(\alpha, R^-)) \setminus \mathcal{M}_b(d(\alpha, R^-))$.

Given $f \in \mathcal{A}(d(0, R^-)) = \sum_{n=0}^{\infty} a_n x^n$, it is well known that $|f(x)|$ has a limit denoted by $|f|(r)$ when $|x|$ tends to r , while being different from r [4]. Then $|f|(r) = \sup_{n \in \mathbb{N}} |a_n| r^n$. This is an absolute value on $\mathcal{A}(d(0, R^-))$ that expands to $\mathcal{M}(d(0, R^-))$.

We call *affine function* a function h of the form $h(x) = ax + b$, $a, b \in K$ and *linear fractional function* a function h of the form $h(x) = \frac{ax + b}{cx + d}$, $a, b, c, d \in K$.

Let us recall some notation concerning the ultrametric Nevanlinna Theory. Let $f \in \mathcal{M}(d(0, R^-))$.

Let $a \in d(0, R^-)$. If a is zero of order q of f , we set $\omega_a(f) = q$. If a is pole of order q of f , we set $\omega_a(f) = -q$. And if a is neither a zero nor a pole of f , we set $\omega_a(f) = 0$.

Let $r \in]0, R[$. Assume that $f(0) \neq 0, \infty$. We denote by $Z(r, f)$ the counting function of zeroes of f in $d(0, r)$ i.e. if (a_n) is the finite or infinite sequence of zeroes of f in $d(0, R^-)$ with respective multiplicity order s_n , we put $Z(r, f) = \sum_{|a_n| \leq r} s_n(\log r - \log |a_n|)$.

In the same way, considering the sequence (b_n) of poles of f in $d(0, r)$ with respective multiplicity order t_n , we put $N(r, f) = \sum_{|b_n| \leq r} t_n(\log r - \log |b_n|)$.

Next, we must define the Nevanlinna function $T(r, f)$ as $\max\{Z(r, f) + \log(|f(0)|), N(r, f)\}$.

Remark: If we change the origin, the functions Z , N , T are not changed, up to an additive constant. Consequently, if we deal with a function f admitting either a zero or a pole at 0, we may perform a change of origin that lets us define the functions Z , N , T : the behaviour of such functions does not depend on the origin we choose.

For each $f \in \mathcal{M}(K)$ (resp. $f \in \mathcal{M}(d(0, R^-))$) we set $\tilde{f}(x) = x^{-\omega_0(f)} f$ and we denote by $\mathcal{M}_f(K)$ (resp. $\mathcal{M}_f(d(0, R^-))$) the set of functions $h \in \mathcal{M}(K)$ (resp. $h \in \mathcal{M}(d(0, R^-))$) such that $T(r, \tilde{h}) = o(T(r, \tilde{f}))$ when r tends to $+\infty$ (resp. when r tends to R). The elements of $\mathcal{M}_f(K)$ (resp. $\mathcal{M}_f(d(0, R^-))$) are called *small functions with regards to f* .

Given $f \in \mathcal{M}(K)$ or $f \in \mathcal{M}_u(d(\alpha, R^-))$, it is well known that there exists at most one value $b \in K$ such that $f(x) \neq b \forall x \in K$ and when such a value b exists, it is then called an *exceptional value of f* (or a *Picard value of f*). Given $f \in \mathcal{A}(K)$ or $f \in \mathcal{A}_u(d(\alpha, R^-))$, there exists no $b \in K$ such that $f(x) \neq b \forall x \in K$. Actually, we know that given $f \in \mathcal{M}(K) \setminus K(x)$ or $f \in \mathcal{M}_u(d(\alpha, R^-))$, there exists at most one value $b \in K$ such that $f(x) - b$ has only finitely many zeroes, and if $f \in \mathcal{A}(K) \setminus K[x]$ (resp. if $f \in \mathcal{A}_u(d(\alpha, R^-))$) then $f - b$ has an infinity of zeroes, for every $b \in K$.

In a previous work [8], the second author gave solutions to the Hayman Conjecture [5], [6] by showing that given a meromorphic function $f \in \mathcal{M}(K)$ or $f \in \mathcal{M}_u(d(\alpha, R^-))$, and $\tau \in K(x)$, then $f' + \tau f^m$ has infinitely many zeroes that are not zeroes of f , whenever $m \geq 5$ and when $m = 1$. Here we mean to look for other results more or less linked to these problems. We shall first examine the field of small functions.

Let us recall the following theorems [1], [2], [3], [7]:

Theorem A: Let $f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathcal{A}(d(0, R^-))$. Let $r \in]0, R^-[$. If f has q zeroes in $d(0, r)$ then $|f|(r) = |a_q| r^q$. Moreover if $f(0) \neq 0$, then $Z(r, f) + \log |f(0)| = \log |f|(r)$.

Theorem B: Let $f \in \mathcal{M}(d(0, R^-))$. For all $r < R$, we have $|f'(r)| \leq \frac{|f|(r)}{r}$. Moreover, $T(r, f)$ is bounded if and only if f belongs to $\mathcal{M}_b(d(0, R^-))$

Theorem C: Let $f \in \mathcal{M}(K)$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Let $P(Y) \in \mathcal{M}_f(K)[Y]$, (resp. $P \in \mathcal{M}_f(d(0, R^-))[Y]$), let $n = \deg(P)$ and assume that f and $P(f)$ have no zero and no pole at 0. Then $T(r, P(f)) = nT(r, f) + o(T(r, f))$ and $T(r, \frac{1}{f}) = T(r, f) + O(1)$. Further, for every $g \in \mathcal{M}(K)$ (resp. $g \in \mathcal{M}(d(0, R^-))$), we have $T(r, P(g)) \leq nT(r, g) + o(T(r, f))$.

By Theorem C we can easily check Theorem 1:

Theorem 1: Let $f \in \mathcal{M}(K)$ (resp. $f \in \mathcal{M}(d(\alpha, R^-))$). Then $\mathcal{M}_f(K)$ (resp. $\mathcal{M}_f(d(\alpha, R^-))$) is a field algebraically closed in $\mathcal{M}(K)$ (resp. in $\mathcal{M}(d(\alpha, R^-))$).

Proof: Let $g \in \mathcal{M}(K)$ (resp. $g \in \mathcal{M}(d(\alpha, R^-))$) and assume that g is algebraic on $\mathcal{M}_f(K)$ (resp. on $\mathcal{M}_f(d(\alpha, R^-))$). Let $P(Y) = \sum_{j=0}^n a_j(x)Y^j \in \mathcal{M}_f(K)[Y]$ (resp. $P(Y) \in \mathcal{M}_f(d(\alpha, R^-))[Y]$) be its minimal polynomial on $\mathcal{M}_f(K)$ (resp. on $\mathcal{M}_f(d(\alpha, R^-))$) (so $a_n = 1$).

Without loss of generality we may assume that $\alpha = 0$ and that none of the a_j have any zero or any pole at 0. Let $Q(Y) = \sum_{j=0}^{n-1} a_j(x)Y^j$. Then $T(r, g^n) = nT(r, g) + O(1)$. On the other hand, and by Theorem C, $T(r, Q(g)) \leq (n-1)T(r, g) + o(T(r, f))$, hence $T(r, g) = o(T(r, f))$, i.e. g lies in $\mathcal{M}_f(K)$ (resp. in $f \in \mathcal{M}_f(d(0, R^-))$).

Similarly, we can show Theorem 2:

Theorem 2: The field $\mathcal{M}_b(K)$ (resp. $\mathcal{M}_b(d(\alpha, R^-))$) is algebraically closed in $\mathcal{M}(K)$ (resp. in $\mathcal{M}(d(\alpha, R^-))$).

We shall notice a property of differential equations of the form $y^{(n)} - \psi y = 0$ that is almost classical.

Theorem 3: Let $\psi \in \mathcal{A}(K)$ (resp. let $\psi \in \mathcal{A}(d(\alpha, R^-))$) and let \mathcal{E} be the differential equations $y^{(n)} - \psi y = 0$. Let E be the sub-vector space of $\mathcal{A}(K)$ (resp. of $\mathcal{A}(d(\alpha, R^-))$) of the solutions of \mathcal{E} . If ψ belongs to $\mathcal{A}(K)$ (resp. if ψ belongs to $\mathcal{A}_u(d(\alpha, R^-))$), then $E = \{0\}$.

If ψ belongs to $\mathcal{A}_b(d(\alpha, R^-))$ and satisfies $\|\psi\|_{d(\alpha, R^-)} > \frac{1}{R^n}$, then $E = \{0\}$.

Proof: In each case, we assume that \mathcal{E} admits a non-identically zero solution h . Then $h^{(n)}$ may not be identically zero.

Suppose first that ψ lies in $\mathcal{A}(K)$. Then $|\psi|(r) = \frac{|h^{(n)}|(r)}{|h|(r)}$ is an increasing function in r in $]0, +\infty[$, a contradiction to the inequality $\frac{|h^{(n)}|(r)}{|h|(r)} \leq \frac{1}{r^n}$ coming from Theorem B.

Now, suppose that $\psi \in \mathcal{A}_u(d(\alpha, R^-))$. Without loss of generality, we may assume $\alpha = 0$. Similarly, $|\psi|(r)$ is unbounded in $]0, R^-]$, a contradiction to $\frac{|h^{(n)}|(r)}{|h|(r)} \leq \frac{1}{r^n}$ again.

In the same way, if ψ belongs to $\mathcal{A}_b(d(0, R^-))$ and satisfies $\|\psi\|_{d(0, R^-)} > \frac{1}{R^n}$, the inequality $\frac{|h^{(n)}|(r)}{|h|(r)} \leq \frac{1}{r^n}$ is then violated when r tends to R .

The problem of a constant Wronskian is involved in several questions.

Theorem 4: *Let $h, l \in \mathcal{A}(K)$ (resp. $h, l \in \mathcal{A}(d(\alpha, R^-))$) and satisfy $h'l - hl' = c \in K$, with h non-affine. If h, l belong to $\mathcal{A}(K)$, then $c = 0$ and $\frac{h}{l}$ is a constant. If $c \neq 0$ and if $h, l \in \mathcal{A}(d(\alpha, R^-))$, there exists $\phi \in \mathcal{A}(d(\alpha, R^-))$ such that $h'' = \phi h$, $l'' = \phi l$. Further, if K has a residue characteristic 0 and if h has at least 2 zeroes in $d(\alpha, R^-)$, then $c = 0$ and $\frac{h}{l}$ is a constant.*

Proof: Suppose $c \neq 0$, if $h(a) = 0$, then $l(a) \neq 0$. Next, h and l satisfy

$$(1) \quad \frac{h''}{h} = \frac{l''}{l}.$$

Remark first that since h is not affine, h'' is not identically zero. Next, every zero of h or l of order ≥ 2 is a trivial zero of $h'l - hl'$. So we can assume that all zeroes of h and l are of order 1.

Now suppose that a zero a of h is not a zero of h'' . Since a is a zero of h of order 1, $\frac{h''}{h}$ has a pole of order 1 at a and so does $\frac{l''}{l}$, hence $l(a) = 0$, a contradiction. Consequently, each zero of h is a zero of order 1 of h and is a zero of h'' . Hence, $\frac{h''}{h}$ is an element ϕ of $\mathcal{M}(K)$ (resp. of $\mathcal{M}(d(\alpha, R^-))$) that has no pole in K (resp. in $d(\alpha, R^-)$) and therefore ϕ lies in $\mathcal{A}(K)$ (resp. in $\mathcal{A}(d(\alpha, R^-))$).

The same holds for l and so, l'' is of the form ψl with $\psi \in \mathcal{A}(K)$ (resp. in $\mathcal{A}(d(\alpha, R^-))$). But since $\frac{h''}{h} = \frac{l''}{l}$, we have $\phi = \psi$.

Now, suppose h, l belong to $\mathcal{A}(K)$. Since h'' is of the form ϕh with $\phi \in \mathcal{A}(K)$, we have $|h''|(r) = |\phi|(r)|h|(r)$. But by Theorem B, we know that $|h''|(r) \leq \frac{1}{r^2}|h|(r)$, a contradiction when r tends to $+\infty$. Consequently, $c = 0$. But then $h'l - hl' = 0$ implies that the derivative of $\frac{h}{l}$ is identically zero, hence $\frac{h}{l}$ is constant.

Suppose now that K has a residue characteristic 0 and h has at least 2 zeroes in $d(\alpha, R^-)$ (taking multiplicity into account) and suppose again that $c \neq 0$. We can find a disk $d(0, s)$ with $s < R$ such that h has $q \geq 2$ zeroes in this disk. Let $t \in]s, R[$ be such that h has q zeroes in $d(0, t)$ and let $h(x) = \sum_{k=0}^{\infty} a_k x^k$. Then by Theorem A we have $|h|(r) = |a_q| r^q \forall r \in [s, t]$ and since K has residue characteristic 0, $|h''|(r) = |a_q| r^{q-2} \forall r \in [s, t]$ and hence $\frac{|h''|(r)}{|h|(r)} = \frac{1}{r^2} \forall r \in [s, t]$. But then, $|\phi|(r) = \frac{1}{r^2} \forall r \in [s, t]$, although $\phi \in \mathcal{A}(d(0, R^-))$, hence $|\phi|(r)$ may not be a decreasing function in r . This finishes proving that $c = 0$ again and therefore $\frac{h}{l}$ is a constant.

Corollary : *Let $h, l \in \mathcal{A}(K)$ with coefficients in \mathbb{Q} , also be entire functions in \mathbb{C} , with h non-affine. If $h'l - hl'$ is a constant c , then $c = 0$.*

Remarks: The entire functions in \mathbb{C} : $h(x) = \cosh x$, $l(x) = \sinh x$ satisfy $h'l - hl' = 1$ but are not entire functions in K : the radius of convergence of both h, l is $p^{\frac{-1}{p-1}}$ when K has residue characteristic $p \neq 0$, is and is 1 when K has residue characteristic 0.

Here we can find again the following result that is known and may be proved without ultrametric properties: *Let F be an algebraically closed field of characteristic zero and let $P, Q \in F[x]$ be such that $PQ' - P'Q$ is a constant c , with $\deg(P) \geq 2$. Then $c = 0$.*

Theorem 5: *Let $\psi \in \mathcal{M}(K)$ (resp. let $\psi \in \mathcal{M}_u(d(\alpha, R^-))$) and let \mathcal{E} be the differential equations $y'' - \psi y = 0$. Let E be the sub-vector space of $\mathcal{A}(K)$ (resp. of $\mathcal{A}(d(\alpha, R^-))$) of the solutions of \mathcal{E} . Then, the dimension of E is 0 or 1.*

Proof: Let $h, l \in E$ be non-identically zero. Then $h''l - hl'' = 0$ and therefore $h'l - hl'$ is a constant c . On the other hand, since h, l are not identically zero, neither are h'', l'' , so h, l are not affine functions.

Suppose first that ψ lies in $\mathcal{M}(K)$. If ψ lies in $\mathcal{A}(K)$, then by Theorem 3, $E = \{0\}$. Now, suppose that ψ lies in $\mathcal{M}(K) \setminus \mathcal{A}(K)$. If $c \neq 0$, by Theorem 4, we have $\psi \in \mathcal{A}(K)$. Consequently, $c = 0$. Therefore $h'l - hl' = 0$ and hence $\frac{h}{l}$ is a constant, hence E is at most of dimension 1.

We now assume that $\psi \in \mathcal{M}_u(d(\alpha, R^-))$. Without loss of generality, we may assume $\alpha = 0$. Suppose $c \neq 0$. By Theorem 4, we can see again that ψ belongs to $\mathcal{A}(d(0, R^-))$, hence to $\mathcal{A}_u(d(0, R^-))$, hence by Theorem 3, $E = \{0\}$. Now, suppose $c = 0$. Thus $\frac{h}{l}$ is a constant, so E is at most of dimension 1.

Remark: The hypothesis ψ unbounded in $d(\alpha, R^-)$ is indispensable to show that the space E is of dimension 0 or 1, as shows the example given again by the p-adic hyperbolic functions $h(x) = \cosh x$ and $l(x) = \sinh x$. The radius of convergence of both h, l is $p^{\frac{-1}{p-1}}$

when K has residue characteristic p and is 1 when K has residue characteristic 0. Of course, both functions are solutions of $y'' - y = 0$ but they are bounded.

Theorem 6: *Let $f \in \mathcal{M}(K)$ be not constant, have no pole of order ≥ 2 and don't let it be a linear fractional function. Then f' has no exceptional value.*

Proof: We can write it $\frac{h}{l}$ with $h, l \in \mathcal{A}(K)$, having no common zero. Since f is not a linear fractional function, at least one of h, l is not an affine function. Since f has no pole of order ≥ 2 , $h'l - hl'$ and l have no common zero, i.e. the zeroes of f' are exactly the zeroes of $h'l - hl'$. Suppose that f' has no zero. Then, neither has $h'l - hl'$ and therefore this is a constant $c \neq 0$, a contradiction by Theorem 4.

Now, suppose f' has an exceptional value b . Then $f' - b$ is the derivative of $f - bx$ whose poles are those of f , as we just saw, $f' - b$ must have at least one zero.

Remark: In Theorem 6, we can't remove the hypothesis *f has no pole of order ≥ 2* , as shows $f(x) = \frac{1}{x^2}$.

Similarly to a theorem in complex analysis, we can show Theorem 7:

Notation: Let $f \in \mathcal{M}(K)$, (resp. $f \in \mathcal{M}(d(0, R^-))$) and let \mathcal{P} be a property satisfied by f at certain points.

Let $r \in]0, R[$. Assume that $f(0) \neq 0, \infty$. We denote by $Z(r, f \mid \mathcal{P})$ the counting function of zeroes of f in $d(0, r)$ at the points where f satisfies \mathcal{P} , i.e. if (a_n) is the finite or infinite sequence of zeroes of f in $d(0, R^-)$ with respective multiplicity order s_n , where \mathcal{P} is satisfied, we put $Z(r, f) = \sum_{|a_n| \leq r, \mathcal{P}} s_n(\log r - \log |a_n|)$.

Theorem 7: *Let $f \in \mathcal{M}(K)$ be transcendental (resp. Let $f \in \mathcal{M}_u(d(\alpha, R^-))$). If there exists $b \in K$ such that $f - b$ has finitely many zeroes, then for every $c \in K^*$, $f' - c$ has infinitely many zeroes.*

Proof: Without loss of generality, we may assume $\alpha = 0$. Let $b \in K$ and suppose that $f - b$ only has a finite number of zeroes. There exist $P \in K[x]$ and $l \in \mathcal{A}(K) \setminus K[x]$ (resp. and $l \in \mathcal{A}_u(d(0, R^-))$) without common zeroes, such that $f = b + \frac{P}{l}$.

Particularly, considering the counting function of zeroes of certain function g whenever $l(x) = 0$ or $l(x) \neq 0$, we shall denote them by $Z(r, g \mid l(x) = 0)$ and $Z(r, g \mid l(x) \neq 0)$, respectively.

Let $c \in K^*$. Remark that $f' - c = \frac{P'l - Pl' - cl^2}{l^2}$. Let a be a zero of l . Then

- (1) $\omega_a((P'l - Pl' - cl^2)) = \omega_a(l) - 1$ due to the fact that $\omega_a(P) \neq 0$. Consequently, if a is a zero of l , it is not a zero of $f' - c$. Else, if a is not a zero of l , then
- (2) $\omega_a(f' - c) = \omega_a(P'l - Pl' - cl^2)$

Consequently, $Z(r, f' - c) = Z(r, (P'l - Pl' - cl^2) \mid l(x) \neq 0)$. But now, by (1) we have

$$(3) \quad Z(r, (P'l - Pl' - cl^2) \mid l(x) = 0) < Z(r, l).$$

and therefore by (2) and (3) we obtain

$$(4) \quad Z(r, f' - c) = Z(r, (P'l - Pl' - cl^2) \mid l(x) \neq 0) > Z(r, P'l - Pl' - cl^2) - Z(r, l)$$

Now, let us examine $Z(r, P'l - Pl' - cl^2)$. Let $r \in]0, +\infty[$ (resp. let $r \in]0, R[$). Since $l \in \mathcal{A}(K)$ is transcendental (resp. $l \in \mathcal{A}_u(d(0, R^-))$), we can check that when r is big enough, we have $|Pl'| (r) < |c| (|l|(r))^2$ and $|Pl|(r) < |c| (|l|(r))^2$, hence clearly $|P'l - Pl'| (r) < |c| (|l|(r))^2$ and hence $|P'l - Pl' - cl^2| (r) = |c| (|l|(r))^2$. Consequently, by Theorem A we have $Z(r, P'l - Pl' - cl^2) = Z(r, l^2) + O(1) = 2Z(r, l) + O(1)$. Therefore by (4) we check that

$$(5) \quad Z(r, f' - c) > Z(r, l).$$

Now, if $l \in \mathcal{A}(K)$, since l is transcendental, by (5), for every $q \in \mathbb{N}$, we have $Z(r, f' - c) > Z(r, l) > q \log r$, when r is big enough, hence $f' - c$ has infinitely many zeroes in K . And similarly if $l \in \mathcal{A}_u(d(0, R^-))$, then by (5), $Z(f' - c)$ is unbounded when r tends to R , hence $f' - c$ has infinitely many zeroes in $d(0, R^-)$.

The following Lemma is useful to prove Theorem 5:

Lemma *Let $f \in \mathcal{M}(K)$ (resp. $f \in \mathcal{M}(d(\alpha, R^-))$) and let $a \in K$ (resp. $a \in d(\alpha, R^-)$) be a zero of $\frac{f'}{f^2} + 1$ that is not a zero of $f' + f^2$. Then a is a pole of order 1 of f and the residue of f at a is 1.*

Proof: Without loss of generality, we assume $\alpha = 0$. We may find $r > |a|$ (resp. $r \in]|a|, R[$) such that $f \in \mathcal{A}(d(0, r^-))$. There exist $h, l \in \mathcal{A}(d(0, r^-))$ without common zeroes such that $f = \frac{h}{l}$.

Since a is not a zero of $f' + f^2$, it is a zero of l and hence it is not a zero of h . Consequently, $h(a)l'(a) - h^2(a) = 0$ and hence $l'(a) = h(a)$ because $h(a) \neq 0$. Thus, a is a pole of order 1 of f and the residue of f at a , of course, is $\frac{h(a)}{l'(a)} = 1$.

Theorem 8 *Let $f \in \mathcal{M}(K)$ be not constant and assume that $\frac{1}{f}$ is not an affine function. Let $b \in K^*$. If $f' + bf^2$ has no zero, then f must admit at least one pole a of order 1 and the residue of f at a is equal to $\frac{1}{b}$.*

Proof: Without loss of generality, up to a change of variable, we may clearly assume that $b = 1$. Suppose that $f' + f^2$ has no zero in K . So, all zeroes of f are of order 1 and hence all poles of $\frac{f'}{f^2} + 1$ are of order 2. Then, for any zero of $\frac{f'}{f^2} + 1$, by the previous Lemma, a is a pole of order 1 of f and the residue of f at a is 1. Each pole of $\frac{f'}{f^2} + 1$ is a zero of f and hence is a pole of order 2 of $\frac{f'}{f^2} + 1$. Consequently, $-\frac{1}{f} + x$ only has poles of order 1.

Suppose that $-\frac{1}{f} + x$ is not a linear fractional function. By Theorem 6, $\frac{f'}{f^2} + 1$ has no exceptional value and therefore it admits a zero a , a contradiction. Since $\frac{1}{f}$ is not an affine function, we may assume $-\frac{1}{f} + x$ to be $\frac{ux+e}{sx+t}$ with $e, t \in K$, $u, s \in K^*$ and $\frac{e}{u} \neq \frac{t}{s}$.

Then,

$$\frac{1}{f} = x - \frac{ux + e}{sx + t} = \frac{sx^2 + tx - ux - e}{sx + t}$$

and, putting $D = sx^2 + tx - ux - e$, we have

$$f' + f^2 = \frac{-sD + (sx + t)(-sx + u)}{D^2}.$$

When the denominator $(sx^2 + x(t - u) - e)^2$ vanishes, we notice that the numerator may not vanish. Indeed, suppose that both have a zero at a point α . So, we have $D(\alpha) = 0 = (s\alpha + t)(-s\alpha + u)$; now, if $-s\alpha + u = 0$, we can derive $s\alpha + t = 0 = u\alpha + e$, hence $ut = es$, a contradiction because $\frac{e}{u} \neq \frac{t}{s}$; and similarly, if $u\alpha + e = 0$, we can derive the same. And since $s \neq 0$, the zeroes of $-sD + (sx + t)(-sx + u)$ do exist. Thus, the zeroes of $-sD + (sx + t)(-sx + u)$ are not zeroes of D and consequently, $f' + f^2$ admits zeroes, which ends the proof.

Remarks: Of course, if $\frac{1}{f}$ is an affine function, $f' + f^2$ has no zeroes, except if it is identically zero. And if it is not identically zero, the residue at the pole is not 1 in the general case.

On a p-adic field, the Hayman Conjecture was solved for $m \geq 5$. Particularly it was shown that $f' + f^m$ admits zeroes that are not zeroes of f for any integer $m \geq 5$ and for $m = 1$. Moreover, it was shown that for $m = 4$, $f' + f^4$ admits at least one zero that is not a zero of f [8]. Here we can see that $f' + f^2$ admits at least one zero provided $\frac{1}{f}$ is neither a constant nor an affine function and f has no pole of order 1 with a residue equal to 1. Thus, if $f \in \mathcal{M}(K)$ has no pole of order 1 with a residue equal to 1 and if $\frac{1}{f}$ is not an affine function, we can say that $f' + f^m$ admits at least one zero for every $m \in \mathbb{N}^*$ except maybe $m = 3$.

In the field \mathbb{C} the classical example of $f(x) = \tan(x)$ shows that a meromorphic function f may be so that $f' + f^2$ admits no zeroes. Precisely, each pole is of order 1 and the residue at each pole is 1. Consequently, in the field K we can ask whether there exist meromorphic functions that only have poles of order 1, with residue 1 at each pole, such that $f' + f^2$ have no zero.

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