

C*-algebras of Penrose's hyperbolic tilings

Hervé Oyono-Oyono, Samuel Petite

▶ **To cite this version:**

Hervé Oyono-Oyono, Samuel Petite. C*-algebras of Penrose's hyperbolic tilings. Journal of Geometry and Physics, 2011, 61, pp.400-424. hal-00383342v2

HAL Id: hal-00383342
https://hal.science/hal-00383342v2

Submitted on 8 Jun 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Copyright

C^*-ALGEBRAS OF PENROSE HYPERBOLIC TILINGS

HERVE OYONO-OYONO AND SAMUEL PETITE

Abstract. Penrose hyperbolic tilings are tilings of the hyperbolic plane which admit, up to affine transformations a finite number of prototiles. In this paper, we give a complete description of the C^*-algebras and of the K-theory for such tilings. Since the continuous hull of these tilings have no transversally invariant measure, these C^*-algebras are traceless. Nevertheless, harmonic currents give rise to 3-cyclic cocycles and we discuss in this setting a higher-order version of the gap-labelling.

Keywords: *Hyperbolic aperiodic tilings, C^*-algebras of dynamical systems, K-theory, Cyclic cohomology*
2000 Mathematics Subject Classification: *37A55, 37B051, 46L55, 46L80*
Subject Classification: *Noncommutative topology and geometry, Dynamical systems*

1. Introduction

The non-commutative geometry of a quasi-periodic tiling studies an appropriate C^*-algebra of a dynamical system (X, G), for a compact metric space X, called *the hull*, endowed with a continuous Lie group G action. This C^*-algebra is of relevance to study the space of leaves which is pathological in any topological sense. The hull owns also a geometrical structure of *lamination* or *foliated space*, the transverse structure being just metric [9]. The C^*-algebras and the non-commutative tools provide then topological and geometrical invariants for the tiling or the lamination. Moreover, some K-theoretical invariants of Euclidean tilings have a physical interpretation. In particular, when the tiling represents a quasi-crystal, the image of the K-theory under the canonical trace labels the gaps in the spectrum of the Schrödinger operator associated with the quasi-crystal [2].

For an Euclidean tiling, the group G is $\mathbb{R}^d$ and $\mathbb{R}^d$-invariant ergodic probability measures on the hull are in one-to-one correspondence with ergodic transversal invariant measures and also with extremal traces on the C^*-algebra [3]. These algebras are well studied and this leads, for instance, to give distinct proofs of the *gap labelling conjecture* [3, 4, 10], i.e. for minimal $\mathbb{R}^d$-action, the image of the K-theory under a trace is the countable subgroup of $\mathbb{R}$ generated by the images under the corresponding transversal invariant measure of the compact-open subsets of the (Cantor) canonical transversal.

For a hyperbolic quasi-periodic tiling, the situation is quite distinct. The group of affine transformations acts on the hull and since this group is not unimodular, there is no transversally invariant measure [16]. A new phenomena shows up for the C^*-algebra of the tiling: it has no trace. Nevertheless, the affine group is amenable, so the hull admits at least one invariant probability measure. These

measures are actually in one-to-one correspondence with harmonic currents [14], and they provide 3-cyclic cocycles on the smooth algebra of the tiling.

The present paper is devoted to give a complete description of the C^*-algebra and the K-theory of a specific family of hyperbolic tilings derivated from the example given by Penrose in [15]. The dynamic of the hulls under investigation, have a structure of double suspension (this make sens in term of groupoids as we shall see in section 5.1) which enables to make explicit computations. This suggests that the pairing with the 3-cyclic cocycle is closely related to the one-dimension gap-labelling for a subshift associated with the tiling. But the right setting to state an analogue of the gap-labelling seems to be Frechet algebras and a natural question is whether this bring in new computable invariants.

Background on tiling spaces is given in the next section and we construct examples of hyperbolic quasi-periodic tilings in the third section. A description of the considered hulls is given in section 4. In section 5, we recall the background on the groupoids and their C^*-algebras. Sections 6 and 7 are devoted to the complete description of the C^*-algebras of the examples and their K-theory groups in terms of generetors are given in section 8. For readers interested in topological invariants of the hull, we compute its K-theory and its Cěch cohomology and we relate these computations to the former one. In the last section we construct 3-cyclic cocycles associated to these tilings and we discuss an odd version of the gap-labelling.

2. Background on tilings

Let $\mathbb{H}_2$ be the real hyperbolic 2-space, identified with the upper half complex plane: $\{(x,y) \in \mathbb{R}^2 \ | y > 0\}$ with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. We denote by G the group of *affine transformations* of this space: i.e. the isometries of $\mathbb{H}_2$ of the kind $z \mapsto az + b$ with a, b reals and $a > 0$.

A tiling $T = \{t_1, \ldots, t_n, \ldots\}$ of $\mathbb{H}_2$, is a collection of convex compact polygons t_i with geodesic borders, called *tiles*, such that their union is the whole space $\mathbb{H}_2$, their interiors are pairwise disjoint and they meet full edge to full edge. For instance, when F is a fundamental domain of a co-compact lattice Γ of isometries of $\mathbb{H}_2$, then $\{\gamma(F),\ \gamma \in \Gamma\}$ is a tiling of $\mathbb{H}_2$. However the set of tilings is much richer than the one given by lattices as we should see later on. Similarly to the Euclidian case, a tiling is said of *G-finite type* or *finite affine type*, if there exists a finite number of polygons $\{p_1, \ldots, p_n\}$ called *prototiles* such that each t_i is the image of one of these polygons by an element of G. Besides its famous Euclidean tiling, Penrose in [15] constructs a finite affine type tiling made with a single prototile which is not stable for any Fuchsian group. The construction goes as follows.

2.1. **Hyperbolic Penrose's tiling.** Let P be the convex polygon with vertices A_p with affix $(p-1)/2 + \imath$ for $1 \leq p \leq 3$ and $A_4 : 2\imath + 1$ and $A_5 : 2\imath$ P is a polygon with 5 geodesic edges. Consider the two maps:

$$R : z \mapsto 2z \text{ and } S : z \mapsto z + 1.$$

The hyperbolic Penrose's tiling is defined by $\mathcal{P} = \{R^k \circ S^n P |\ n, k \in \mathbb{Z}\}$ (see figure 1). This is an example of finite affine type tiling of $\mathbb{H}_2$.

This tiling is stable under no co-compact group of hyperbolic isometries. The proof is homological: we associate with the edge A_4A_5 a positive charge and two negative charges with edges A_1A_2, A_2A_3. If $\mathcal{P}$ was stable for a Fuchsian group, then P would tile a compact surface. Since the edge A_4A_5 can meet only the edges

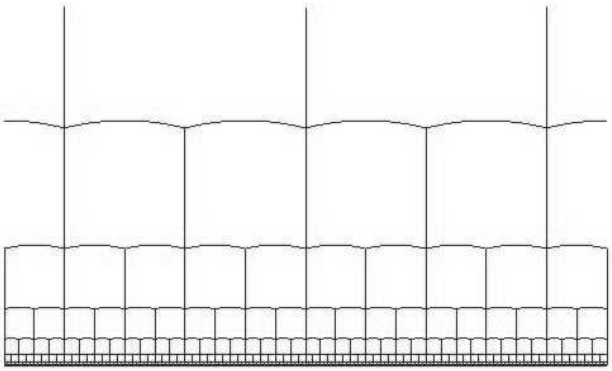


FIGURE 1. The hyperbolic Penrose's tiling

A_1A_2 or A_2A_3, the surface has a neutral charge. This is in contradiction with the fact P is negatively charged.
G. Margulis and S. Mozes [13] have generalized this construction to build a family of prototiles which cannot be used to tile a compact surface. Notice the group of isometries which preserves $\mathcal{P}$ is not trivial and is generated by the transformation R. In order to break this symmetry, it is possible, by a standard way, to decorate prototiles to get a new finite affine type tiling which is stable under no non-trivial isometry (we say in this case that the tiling is *aperiodic*).

3. Background on tiling spaces

In this section, we recall some basic definitions and properties on dynamical systems associated with tilings. We refer to [5], [12] and [21] for the proofs. We give then a description of the dynamical system associated to the hyperbolic Penrose's tiling.

3.1. **Action on tilings space.** First, note that the group G acts transitively, freely (without a fixed point) and preserving the orientation of the surface $\mathbb{H}_2$, thus G is a Lie group homeomorphic to $\mathbb{H}_2$. The metric on $\mathbb{H}_2$ gives a left multiplicative invariant metric on G. We fix the point O in $\mathbb{H}_2$ with affix i that we call *origin*.
For a tiling T of G finite type and an isometry p in G, the image of T by p is again a tiling of $\mathbb{H}_2$ of G finite type. We denote by $G.T$ the set of tilings which are image of T by isometries in G. The group G acts on this set by the left action:

$$\begin{array}{ccc} G \times G.T & \longrightarrow & G.T \\ (p, T') & \longmapsto & p.T' = p(T'). \end{array}$$

We equip $G.T$ with a metrizable topology, so that the action becomes continuous. A base of neighborhoods is defined as follows: two tilings are close one to the other if they agree, on a big ball of $\mathbb{H}_2$ centered at the origin, up to an isometry in G close to the identity. This topology can be generated by the metric δ on $G.T$ defined by (see [5]):
For T and T' be two tilings of $G.T$, let

$$A = \{\epsilon \in (0, \frac{1}{\sqrt{2}}] \mid \exists\ g \in B_\epsilon(Id) \subset G \text{ s.t. } g.T \cap B_{1/\epsilon} = T' \cap B_{1/\epsilon}\}$$

where $B_{1/\epsilon}$ is the set of points $x \in \mathbb{H}_2 \cong G$ such that $d(x, O) < 1/\epsilon$.
We define:

$$\delta(T, T') = \inf A \text{ if } A \neq \emptyset$$

$$\delta(T, T') = \frac{1}{\sqrt{2}} \text{ else.}$$

The *continuous hull* of the tiling T, is the metric completion of $G.T$ for the metric δ. We denote it by X_T^G. Actually this space is a set of tilings of $\mathbb{H}_2$ of G-finite type. A *patch* of a tiling T is a finite set of tiles of T. It is straightforward to show that patches of tilings in X_T^G are copies of patches of T. The set X_T^G is then a compact metric set and the action of G on $G \cdot T$ can be extended to a continuous left action on this space. The dynamical system (X_T^G, G) has a dense orbit: the orbit of T.
Some combinatorial properties can be interpreted in a dynamical way like, for instance, the following.

Definition 3.1. *A tiling T satisfies the repetitivity condition if for each patch P, there exists a real $R(P)$ such that every ball of $\mathbb{H}_2$ with radius $R(P)$ intersected with the tiling T contains a translated by an element G of the patch P.*

This definition can be interpreted from a dynamical point of view (for a proof see for instance [8]).

Proposition 3.2 (Gottschalk)**.** *The dynamical system (X_T^G, G) is minimal (any orbit is dense) if and only if the tiling T satisfies the repetitivity condition.*

We call a tiling *aperiodic* if the action of G on X_T^G is free: for all $p \neq Id$ of G and all tilings T' of X_T^G we have $p.T' \neq T'$.
As we have seen in the former section the hyperbolic Penrose's tiling is not aperiodic, however, using this example, we shall construct in section 4 uncountably many examples of repetitive and aperiodic affine finite type tilings.

When the tiling T is aperiodic and repetitive, the hull X_T^G has also a geometric structure of a specific lamination called a *G-solenoid* (see [5]). Locally at any point x, there exists a *vertical germ* which is a Cantor set included in X_T^G, transverse to the local G-action and which is defined independently of the neighborhood of the point x. This implies that X_T^G is locally homeomorphic to the Cartesian product of a Cantor set with an open subset (called a *slice*) of the Lie group G. The connected component of the slices that intersect is called a *leaf* and has a manifold structure. Globally, X_T^G is a disjoint union of uncountably many leaves, and it turns out that each leaf is a G-orbit. Since the action is free, each leaf is homeomorphic to $\mathbb{H}_2$.

In the aperiodic case, the G-action is *expansive*: There exists a positive real ϵ such that for every points T_1 and T_2 in the same vertical in X_T^G, if $\delta(T_1.g, T_2.g) < \epsilon$ for every $g \in G$, then $T_1 = T_2$.
Furthermore this action has locally constant return times: if an orbit (or a leaf) intersects two verticals V and V' at points v and $v.g$ where $g \in G$, then for any point w of V close enough to v, $w.g$ belongs to V'.

3.2. **Structure of the hull of the Penrose Hyperbolic tilings.** First recall the notion of suspension action for X a compact metric space and $f : X \to X$ a homeomorphim. The group $\mathbb{Z}$ acts diagonally on the product space $X \times \mathbb{R}$ by the

following homeomorphism denoted $\mathcal{A}_f$

$$\begin{aligned} \mathcal{A}_f :\ & X \times \mathbb{R} \to X \times \mathbb{R} \\ & (x,t) \mapsto (f(x), t-1) \end{aligned}$$

The quotient space of $(X \times \mathbb{R})/\mathcal{A}_f$, where two points are identified if they belong to the same orbit, is a compact set for the product topology and is called *the suspension* of (X, f). The group $\mathbb{R}$ acts also diagonally on $X \times \mathbb{R}$: trivially on X and by translation on $\mathbb{R}$. Since this action commutes with $\mathcal{A}_f$, this induces a continuous $\mathbb{R}$ action on the suspension space $(X \times \mathbb{R})/\mathcal{A}_f$ that we call: the *suspension action* of the system (X, f) and we denote it by $\big((X \times \mathbb{R})/\mathcal{A}_f, \mathbb{R}\big)$.

We recall here, the construction of the dyadic completion of the integers. On the set of integers $\mathbb{Z}$, we consider the dyadic norm defined by

$$|n|_2 = 2^{-\sup\{p \in \mathbb{N},\ 2^p \text{ divides } |n|\}} \qquad n \in \mathbb{Z}.$$

Let Ω be the completion of the set $\mathbb{Z}$ for the metric given by $|.|_2$. The set Ω has a commutative group structure where $\mathbb{Z}$ is a dense subgroup, and Ω is a Cantor set. The continuous action given by the map $o :\ x \mapsto x + 1$ on Ω is called *adding-machine* or *odometer* and is known to be minimal and equicontinuous. We denote by $((\Omega \times \mathbb{R})/\mathcal{A}_o, \mathbb{R})$ the suspension action of this homeomorphism.

Recall that a conjugacy map between two dynamical systems is a homeomophism which commutes with the actions. Let $\mathcal{N}$ be the group of transformations $\{z \mapsto z + t,\ t \in \mathbb{R}\}$ isomorphic to $\mathbb{R}$.

Proposition 3.3. *Let $X_{\mathcal{P}}^{\mathcal{N}}$ be the closure (for the tiling topology) of the orbit $\mathcal{N}.\mathcal{P} \subset X_{\mathcal{P}}^{G}$. Then the dynamical system $(X_{\mathcal{P}}^{\mathcal{N}}, \mathcal{N})$ is conjugate to the suspension action of the odometer $((\Omega \times \mathbb{R})/\mathcal{A}_o, \mathbb{R})$.*

Proof. Let $\phi : \mathcal{N}.\mathcal{P} \to (\Omega \times \mathbb{R})/\mathcal{A}_o$ be the map defined by $\phi(\mathcal{P}+t) = [0,t]$ where $[0,t]$ is the $\mathcal{A}_o$-class of $(0,t) \in \Omega \times \mathbb{R}$. Since the tiling is invariant under no translations, the application ϕ is well defined. It is straightforward to check that ϕ is continuous for the tiling topology and for the topology on $(\Omega \times \mathbb{R})/\mathcal{A}_o$ arising from the dyadic topology on Ω. So the map ϕ extends by continuity to $X_{\mathcal{P}}^{\mathcal{N}}$. Let us check that ϕ is a homeomorphism by constructing its inverse. Let $\psi : \mathbb{Z} \times \mathbb{R} \to X_{\mathcal{P}}^{\mathcal{N}}$ defined by $\psi(n,t) = \mathcal{P} + n + t$. This application is continuous for the dyadic topology, so it extends by continuity to $\Omega \times \mathbb{R}$. Notice $\psi(n,t)$ is constant along the orbits of the $\mathcal{A}_o$ action on $\mathbb{Z} \times \mathbb{R}$ which is dense in $\Omega \times \mathbb{R}$. Thus ψ is constant along the $\mathcal{A}_o$-orbits in $\Omega \times \mathbb{R}$ and ψ factorizes through a map $\overline{\psi}$ from the suspension $(\Omega \times \mathbb{R})/\mathcal{A}_o$ to $X_{\mathcal{P}}^{\mathcal{N}}$. It is plain to check that $\overline{\psi} \circ \phi = Id$ on the dense set $\mathcal{N}.\mathcal{P}$ and that $\phi \circ \overline{\psi} = Id$ on the dense set $\pi(\mathbb{Z} \times \mathbb{R})$ where $\pi : \Omega \times \mathbb{R} \to (\Omega \times \mathbb{R})/\mathcal{A}_o$ denotes the canonical projection. Hence ϕ is an homeomorphism from $X_{\mathcal{P}}^{\mathcal{N}}$ onto $(\Omega \times \mathbb{R})/\mathcal{A}_o$. It is obvious that ϕ commutes with the $\mathbb{R}$-actions. □

4. Examples

We construct in this section a family of tilings of $\mathbb{H}_2$ of finite affine type, indexed by sequences on a finite alphabet. For uncountably many of them, the tilings will be aperiodic and repetitive, the action on the associated hull will be free and minimal. A description of these actions in terms of double-suspension is given.

4.1. **Construction of the examples.** To construct such tilings we will use the hyperbolic Penrose's tiling described in section 2.1, so we will keep the notations of this section. Recall that its stabilizer group under the action of G, is the group $\langle\, R \,\rangle$ generated by the affine transformations R. The main idea is to "decorate" this tiling in order to break its symmetry, the decoration will be coded by a sequence on a finite alphabet. By a decoration, we mean that we will substitute to each tile t the same polygon t equipped with a color. We take the convention that two colored polygons are the same if and only if the polygons are the same up to an affine map and they share the same color. By substituting each tile by a colored tile, we obtain a new tiling of finite affine type with a bigger number of prototiles. Notice that the coloration is not canonical. It also possible to do the same by substituting to a tile t, an unique finite family of convex tiles $\{t_i\}_i$, like triangles, such that the union of the t_i is t and the tiles t_i overlaps only on their borders. We choose the coloration only for presentation reasons.

Let r be an integer bigger than 1. We associate to each element of $\{1, \ldots, r\}$ an unique color. Let P be the polygon defined in section 2.1 to construct the Penrose's tiling. For an element i of $\{1, \ldots, r\}$, we denote by P_i the prototile P colored in the color i. To a sequence $w = (w_k)_k \in \{1, \ldots, r\}^{\mathbb{Z}}$, we associate the G-finite-type tiling $\mathcal{P}(w)$ built with the prototiles P_i for i in $\{1, \ldots, r\}$ and defined by:

$$\mathcal{P}(w) = \{R^q \circ S^n(P_{w_{-q}}),\ n, q \in \mathbb{Z}\}.$$

Notice that the stabilizer of this tiling is a subgroup of $\langle\, R \,\rangle$.

The set of sequences on $\{1, \ldots, r\}$ is the product space $\{1, \ldots, r\}^{\mathbb{Z}}$ which is a Cantor set for the product topology. There exists a natural homeomorphism on it called the *shift*. To a sequence $(w_n)_{n\in\mathbb{Z}}$ the shift σ associates the sequence $(w_{n+1})_{n\in\mathbb{Z}}$. Let Z_w denote the closure of the orbit of w by the action of the shift σ: $Z_w = \overline{\{\sigma^n(w),\ n \in \mathbb{Z}\}}$. The set Z_w is a compact metric space stable under the action of σ.

Remark 4.1. *The map*

$$Z_\omega \to X^G_{\mathcal{P}(w)};\ \omega' \mapsto \mathcal{P}(w')$$

is continuous.

Since $R.\mathcal{P}(w)$ denotes the tiling image of $\mathcal{P}(w)$ by R, we get the relation

$$R.\mathcal{P}(w) = \mathcal{P}(\sigma(w)). \tag{4.1}$$

Thanks to this, we obtain the following property:

Lemma 4.2.

- *The sequence w is aperiodic for the shift-action, if and only if $\mathcal{P}(w)$ is stable under no non-trivial affine map.*
- *The dynamical system (Z_w, σ) is minimal, if and only if $(X^G_{\mathcal{P}(w)}, G)$ is minimal.*

Proof. The first point comes from the relation 4.1 and from the fact that the stabilizer of $\mathcal{P}(w)$ is a subgroup of $\langle\, R \,\rangle$. The last point comes from the characterization of minimal sequences: (Z_w, σ) is minimal if and only if each words in w appears infinitely many times with uniformly bounded gap [19]. This condition is equivalent to the repetitivity of $\mathcal{P}(w)$. □

Recall that we have defined the group $\mathcal{N} = \{z \mapsto z + t,\ t \in \mathbb{R}\}$ and that $X_{\mathcal{P}}^{\mathcal{N}}$ stands for the closure (for the tiling topology) of the $\mathcal{N}$-orbit $\mathcal{N}.\mathcal{P}$ in $X_{\mathcal{P}}^{G}$ of the uncolored tiling $\mathcal{P}$. Notice that the continuous action of R on $X_{\mathcal{P}}^{G}$ preserves the orbit $\mathcal{N}.\mathcal{P}$ so we get an homeomorphism of $X_{\mathcal{P}}^{\mathcal{N}}$ that we denote also by R. We consider on the space $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$ equipped with the product topology, the homeomorphism $\mathcal{R}$ defined by $\mathcal{R}(\mathcal{T}, w', t) = (R.\mathcal{T}, \sigma(w'), t/2)$. The quotient space $(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$, where the points in the same $\mathcal{R}$ orbit are identified, is a compact space.

The affine group G also acts on the left on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$; where the action of an element $g : z \mapsto az + b$ is given by the homeomorphism

$$(\mathcal{T}, w', t) \mapsto (\mathcal{T} + \frac{b}{at}, w', at) = g.(\mathcal{T}, w', t).$$

It is straightforward to check that the application $\mathcal{R}$ commutes with this action, so this defines a G-continuous action on the quotient space $(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$.

Proposition 4.3. *Let w be an element of $\{1, \dots, r\}^{\mathbb{Z}}$. Then the map*

$$\begin{array}{rcl} \Psi : G.\mathcal{P}(w) & \to & (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R} \\ g.\mathcal{P}(w) & \mapsto & [g.(\mathcal{P}, w, 1)] \end{array}$$

where $[x]$ denotes the $\mathcal{R}$-class of x, extends to a conjugacy map between $(X_{\mathcal{P}(w)}^{G}, G)$ and $\big((X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R},\ G\big)$.

Proof. Let Φ be the transformation $\mathcal{N}.\mathcal{P} \times Z_w \times \mathbb{R}_+^* \to X_{\mathcal{P}(w)}^{G}$ defined by

$$\Phi(\mathcal{P} + \tau, w', t) = R_t.(\mathcal{P}(w') + \tau)$$

where R_t denotes the map $z \mapsto tz$. According to remark 4.1, the application Φ is continuous for the tiling topology on $\mathcal{N}.\mathcal{P}$, so it extends by continuity to a continuous map from $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$ to $X_{\mathcal{P}(w)}^{G}$. Thanks to relation (4.1), we get $\Phi \circ \mathcal{R} = \Phi$ on the dense subset $\mathcal{N}.\mathcal{P} \times Z_w \times \mathbb{R}_+^*$. Therefore the map Φ factorizes throught a continuous map $\overline{\Phi} : (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R} \to X_{\mathcal{P}(w)}^{G}$. Since the stabilizer of the tiling $\mathcal{P}(w)$ is a subgroup of the one generated by the transformation R, and by relation (4.1), the map

$$\Psi : G \cdot \mathcal{P}(w) \to \big(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*\big)/\mathcal{R};\ R_t \cdot \mathcal{P}(w) + \tau \mapsto [\mathcal{P} + \tau/t, \omega, t]$$

is well defined. It is straightforward to check that Ψ is continuous, so it extends to a continuous map from $X_{\mathcal{P}(w)}^{G}$ to $\big(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*\big)/\mathcal{R}$ that we denote again Ψ. Furthermore we have $\overline{\Phi} \circ \Psi = Id$ on $G \cdot \mathcal{P}(w)$ and $\Psi \circ \overline{\Phi} = Id$ on the dense set $\pi(\mathcal{N}.\mathcal{P} \times Z_w \times \mathbb{R}_+^*)$ where π denotes the canonical projection $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^* \to \big(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*\big)/\mathcal{R}$. Hence the map $\overline{\Phi}$ is an homeomorphism. The homeomorphism Ψ obviously commutes with the action. □

Notice that, $X_{\mathcal{P}}^{\mathcal{N}}$ is locally the Cartesian product of a Cantor set by an interval of $\mathbb{R}$. For $w \in \{1, \dots, r\}^{\mathbb{Z}}$, $X_{\mathcal{P}(w)}^{G}$ is locally homeomorphic the product of a Cantor set by an open subset (a slice) of $\mathbb{R}_+^* \times \mathbb{R}$ since the Cartesian product of two Cantor sets is again a Cantor set. The G-action maps slices onto slices.

4.2. **Ergodic properties of Penrose's tilings.** For a metric space X and a continuous action of a group Γ on it, a Γ-*invariant measure* is a measure μ defined on the Borel σ-algebra of X which is invariant under the action of Γ i.e.: For any measurable set $B \subset X$ and any $g \in \Gamma$, $\mu(B.g) = \mu(B)$. For instance, any group Γ acts on itself by right multiplication, there exists (up to a scalar) only one measure invariant for this action: it is called the *Haar* measure.

Any action of an amenable group Γ (like $\mathbb{Z}$, $\mathbb{R}$ and all their extensions) on a compact metric space X admits a finite invariant measure and in particular, any homeomorphism f of X preserves a probability measure. An *ergodic* invariant measure μ is such that every measurable functions constant along the orbits are μ almost surely constant. Every invariant measure is the sum of ergodic invariant measures [19]. A conjugacy map sends the invariant measure to invariant measure and the ergodic measures to the ergodic measures.

In our case, the group of affine transformations G, is the extension of two groups isomorphic to $\mathbb{R}$, hence is amenable. It is well known that the only invariant measures for the suspension action $\big((X \times \mathbb{R})/\mathcal{A}_f, \mathbb{R}\big)$ are locally the images through the canonical projection $\pi : X \times \mathbb{R} \to \big(X \times \mathbb{R}\big)/\mathcal{A}_f$ of the measures $\mu \otimes \lambda$ where μ is a f-invariant measure on X and λ denotes the Lebesgue measure of $\mathbb{R}$. The proof is actually contained in property 4.4.

It is well known also that the map $o : x \mapsto x + 1$ on the dyadic set of integers Ω, admits only one invariant probability measure: the Haar probability measure on Ω. Hence the suspension of this action $\big((\Omega \times \mathbb{R})/\mathcal{A}_o, \mathbb{R}\big)$ admits only one invariant probability measure. By proposition 3.3, $X_{\mathcal{P}}^{\mathcal{N}}$ has only one invariant probability measure ν. Notice that the map R preserves $X_{\mathcal{P}}^{\mathcal{N}}$, and since $R\mathcal{N}R^{-1} = \mathcal{N}$, the probability $R_*\nu$ is $\mathcal{N}$-invariant and hence R preserves ν. Nevertheless, the local product decomposition of ν is not invariant by R, because R divides by 2 the length of the intervals of the $\mathcal{N}$-orbit. So R has to inflate the Haar measure on Ω by a factor 2.

Proposition 4.4. *If w is an element in $\{1, \dots, r\}^{\mathbb{Z}}$, then any finite invariant measure of $\big((X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R},\ G\big)$ is locally the image through the projection $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^* \to (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$ of a measure $\nu \otimes \mu \otimes m$ where*

- *ν is the only invariant probability measure of $(X_{\mathcal{P}}^{\mathcal{N}}, \mathbb{R})$;*
- *m is the Haar measure of $(\mathbb{R}_+^*, .)$;*
- *μ is a finite invariant measure of (Z_w, σ).*

Proof. It is enough to prove this for an ergodic finite G-invariant measure $\overline{\theta}$ on the suspension $(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$. Since $\mathcal{R}$ acts cocompactly on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$, $\overline{\theta}$ defines a finite measure on a fundamental domain of $\mathcal{R}$, and the sum of all the images of this measure by iterates of $\mathcal{R}$ and $\mathcal{R}^{-1}$ defines a σ-finite measure θ on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$ which is G and $\mathcal{R}$-invariant.
Let $\pi_2 : X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^* \to Z_w$ be the projection to the second coordinate, then $\pi_{2*}\theta$ is a shift invariant measure on Z_w. The measure θ can be disintegrated over $\pi_{2*}\theta = \mu$ by a family of measures $(\lambda_{w'})_{w' \in Z_w}$ defined for μ-almost every $w' \in Z_w$ on $X_{\mathcal{P}}^{\mathcal{N}} \times \{w'\} \times \mathbb{R}_+^*$ such that

$$\theta(B \times C) = \int_C \lambda_{w'}(B) d\mu(w'),$$

for any Borel sets $B \subset X_{\mathcal{P}}^{\mathcal{N}} \times \mathbb{R}_+^*$ and $C \subset Z_w$.

The G-invariance of θ implies that the measures $\lambda_{w'}$ are G-invariant for almost all w'. The projection to the first coordinate $\pi_1 : X_{\mathcal{P}}^{\mathcal{N}} \times \{w'\} \times \mathbb{R}_+^* \to X_{\mathcal{P}}^{\mathcal{N}}$ is $\mathcal{N}$-equivariant. The measures $\pi_{1*}\lambda_{w'}$ are then $\mathcal{N}$-invariant measures, hence are proportional to ν. Each measure $\lambda_{w'}$ can be disintegrated over ν by a family of measures $(m_{x,w'})_{x\in X_{\mathcal{P}}^{\mathcal{N}},\, w'\in Z_w}$ on $\mathbb{R}_+^*$ defined for ν-almost every $x \in X_{\mathcal{P}}^{\mathcal{N}}$, so that

$$\lambda_{w'}(B \times \{w'\} \times I) = \int_B \int_I m_{x,w'} d\nu(x),$$

for any Borel sets $B \subset X_{\mathcal{P}}^{\mathcal{N}}$ and $I \subset \mathbb{R}_+^*$. Each measure $\lambda_{w'}$ is invariant under the action of transformations of the kind $z \mapsto az$ for $a \in \mathbb{R}_+^*$. It is then straightforward to check that the measures $m_{x,w'}$ are multiplication-invariant for almost every x. By unicity of the Haar measure, there exists a measurable positive function $(x, w') \mapsto h(x, w')$ defined almost everywhere so that $m_{x,w'} = h(x, w')m$. The $\mathcal{N}$-invariance of the measures $\lambda_{w'}$ implies that the map h is almost surely constant along the $\mathcal{N}$-orbits, and the $\mathcal{R}$-invariance of θ implies that h is almost surely constant along the $\mathcal{R}$-orbits. This defines then a measurable map on the quotient space by $\mathcal{R}$ which is G-invariant, the ergodicity of $\overline{\theta}$ implies this map is almost surely constant. □

Notice that an invariant measure on $X_{\mathcal{P}(w)}^G$ can be decomposed locally into the product of a measure on a Cantor set by a measure along the leaves. Since the map R does not preserve the transversal measure on Ω in $X_{\mathcal{P}}^{\mathcal{N}}$, the holonomy groupoid of $X_{\mathcal{P}(w)}^G$ does not preserve the transversal measure on the Cantor set.
The G-action is locally free and acts transitively on each leaf, so each orbits inherits a hyperbolic 2-manifolds structure. Actually, $X_{\mathcal{P}(w)}^G$ can be equipped with a continuous metric with a constant curvature -1 in restriction to the leaves. The invariant measures of $X_{\mathcal{P}(w)}^G$ have then also a geometric interpretation in terms of *harmonic* measures, a notion introduced by L. Garnett in [7].

Definition 4.5. *A probability measure μ on M is harmonic if*

$$\int_M \Delta f d\mu = 0$$

for any continuous function f with restriction to leaves of class $\mathcal{C}^2$, where Δ denotes the Laplace-Beltrami operator in restriction on each leaf.

Actually, it is shown in [14] , that on $X_{\mathcal{P}(w)}^G$ the notions of harmonic and invariant measures are the same and such measures can be described in terms of inverse limit of vectoriel cones.

5. Transformation groupoids

We gather this section with results on groupoids and their C^*-algebras. Good material on this topic can be found in [20]. Let us fix first some notations. Let $\mathcal{G}$ be a locally compact groupoid, with base space X, range and source maps respectively $r : \mathcal{G} \to X$ and $s : \mathcal{G} \to X$. Recall that X can be viewed as a closed subset of $\mathcal{G}$ (the set of units). For any element x of X, we set

$$\mathcal{G}^x = \{\gamma \in \mathcal{G} \text{ such that } r(\gamma) = x\}$$

and

$$\mathcal{G}_x = \{\gamma \in \mathcal{G} \text{ such that } s(\gamma) = x\}.$$

Let us denote for any γ in $\mathcal{G}$ by $L_\gamma : \mathcal{G}^{s(\gamma)} \to \mathcal{G}^{r(\gamma)}$ the left translation by γ. Thourought this section, all the groupoids will be assumed locally compact and second countable. Recall that a Haar system λ for $\mathcal{G}$ is a family $(\lambda^x)_{x\in X}$ of borelian measures on $\mathcal{G}$ such that

(1) the support of λ^x is $\mathcal{G}^x$;
(2) for any f in $C_c(\mathcal{G})$, the map $X \to \mathbb{C}$; $x \to \int_{\mathcal{G}^x} f d\lambda^x$ is continuous;
(3) $L_{\gamma_*}\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$ for all γ in Γ.

Our prominent examples of groupoid will be *semi-direct product groupoid*: let H be a locally compact group acting on a locally compact space X. The semi-direct product groupoid $X \rtimes H$ of X by H is defined by

- $X \times H$ as a topological space ;
- the base space is X and the structure maps are $r : X \rtimes H \to X$; $(x, h) \mapsto x$ and $s : X \rtimes H \to X$; $(x, h) \mapsto h^{-1}x$;
- the product is $(x, h) \cdot (h^{-1}x, h') = (x, hh')$ for x in X and h and h' in H.

Let μ be a left Haar mesure on H. Then the groupoid $X \rtimes H$ is equipped with a Haar system $\lambda^\mu = (\lambda^\mu_x)_{x\in X}$ given for any f in $C_c(X \times H)$ and any x in X by $\lambda^\mu_x(f) = \int_H f(x, h)d\mu(h)$.

5.1. **Suspension of a groupoid.** Recall that any automorphism α of a groupoid $\mathcal{G}$ induces a homeomorphism of its base space X that we shall denote by α_X.

Definition 5.1. *Let $\mathcal{G}$ be a groupoid with base space X equipped with a Haar system $\lambda = (\lambda^x)_{x\in X}$. A groupoid automorphism $\alpha : \mathcal{G} \to \mathcal{G}$ is said to preserve the Haar system λ if there exists a continuous function $\rho_\alpha : \mathcal{G} \to \mathbb{R}^+$ such that for any x in X the measures $\alpha_*\lambda^x$ and $\lambda^{\alpha(x)}$ on $\mathcal{G}^{\alpha(x)}$ are in the same class and ρ_α restricted to $\mathcal{G}^{\alpha(x)}$ is $\frac{d\alpha_*\lambda^x}{d\lambda^{\alpha(x)}}$. The map ρ_α is called the density of α.*

Remark 5.2. *Let $\mathcal{G}$ be a groupoid with base space X and Haar system $\lambda = (\lambda^x)_{x\in X}$ and let $\alpha : \mathcal{G} \to \mathcal{G}$ be an automorphism of groupoid preserving the Haar system λ.*

(1) *Since $L_\gamma \circ \alpha = \alpha \circ L_{\alpha^{-1}(\gamma)}$ for any γ in $\mathcal{G}$, we get that*

$$L_{\gamma,*}\alpha_*\lambda^{\alpha^{-1}(s(\gamma))} = \alpha_* L_{\alpha^{-1}(\gamma),*}\lambda^{s(\alpha^{-1}(\gamma))} = \alpha_*\lambda^{r(\alpha^{-1}(\gamma))}.$$

Since $L_{\gamma,*}\alpha_*\lambda^{\alpha^{-1}(s(\gamma))}$ is a measure on $\mathcal{G}^{r(\gamma)}$ absolutly continuous with respect to $L_{\gamma,*}\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$ with density $\rho_\alpha \circ L_{\gamma^{-1}}$ we see that $\rho_\alpha \circ L_{\gamma^{-1}}$ and ρ_α coincide on $\mathcal{G}^{r(\gamma)}$. In particular ρ_α is constant on $\mathcal{G}_x$ for any x in X.

(2) *The automorphism of groupoid $\alpha^{-1} : \mathcal{G} \to \mathcal{G}$ also preserves the Haar system λ and $\rho_{\alpha^{-1}} = 1/\rho_\alpha \circ \alpha$.*

Definition 5.3. *Let $\mathcal{G}$ be a groupoid with base space X, range and source map r and s and let $\alpha : \mathcal{G} \to \mathcal{G}$ be a groupoid automorphism. Using the notations of section 3.2 the suspension of the groupoid $\mathcal{G}$ respectively to α is the groupoid $\mathcal{G}_\alpha \overset{def}{=\!=} (\mathcal{G} \times \mathbb{R})/\mathcal{A}_\alpha$ with base space $X_\alpha \overset{def}{=\!=} (X \times \mathbb{R})/\mathcal{A}_{\alpha_X}$. For any γ in $\mathcal{G}$ and t in $\mathbb{R}$, let us denote by $[\gamma, t]$ the class of (γ, t) in $\mathcal{G}_\alpha$.*

- *The range map r_α and the source map s_α are defined in the following way:*
  - *$r_\alpha([\gamma, t]) = [r(\gamma), t]$ for every γ in $\mathcal{G}$ and t in $\mathbb{R}$;*
  - *$s_\alpha([\gamma, t]) = [s(\gamma), t]$ for every γ in $\mathcal{G}$ and t in $\mathbb{R}$;*
- *Let γ and γ' be elements of $\mathcal{G}$ such that $s(\gamma) = r(\gamma')$ and let t be in $\mathbb{R}$, then $[\gamma, t] \circ [\gamma', t] = [\gamma \circ \gamma', t]$;*

- $[\gamma, t]^{-1} = [\gamma^{-1}, t]$.

There is an action of $\mathbb{R}$ *on* $\mathcal{G}_\alpha$ *by automorphisms given for* s *in* $\mathbb{R}$ *and* $[\gamma, t]$ *in* $\mathcal{G}_\alpha$ *by* $s \cdot [\gamma, t] = [\gamma, t + s]$.

Lemma 5.4. *Let* $\mathcal{G}$ *be a groupoid with base space* X *equipped with a Haar system* $\lambda = (\lambda^x)_{x\in X}$ *and let* $\alpha : \mathcal{G} \to \mathcal{G}$ *be an automorphism preserving the Haar System* λ. *Let us assume that* $\rho_\alpha = \rho_\alpha \circ \alpha$. *Then* $\mathcal{G}_\alpha$ *admits a Haar system* $\lambda_\alpha = \left(\lambda_\alpha^{[x,t]}\right)_{[x,t]\in X_\alpha}$ *given for any* $[x, t]$ *in* X_α *and any continuous fonction* f *in* $C_c\left(\mathcal{G}_\alpha^{[x,t]}\right)$ *by*

$$\lambda_\alpha^{[x,t]}(f) = \int_{\mathcal{G}^x} \rho_\alpha(\gamma)^{-t} f([\gamma, t]) d\lambda^x(\gamma).$$

Proof.

- Let us prove first that the definition of $\lambda_\alpha^{[x,t]}(f)$ for $[x, t]$ in X_α and f in $C_c\left(\mathcal{G}_\alpha^{[x,t]}\right)$ makes sense.

$$\begin{aligned}\int_{\mathcal{G}^x} \rho_\alpha(\gamma)^{-t} f([\gamma, t]) d\lambda^x(\gamma) &= \int_{\mathcal{G}^x} \rho_\alpha(\alpha(\gamma))^{-t} f([\alpha^{-1}(\alpha(\gamma), t]) d\lambda^x(\gamma) \\ &= \int_{\mathcal{G}^{\alpha(x)}} \rho_\alpha(\gamma)^{-t+1} f([\alpha^{-1}(\gamma), t]) d\lambda^{\alpha(x)}(\gamma) \\ &= \int_{\mathcal{G}^{\alpha(x)}} \rho_\alpha(\gamma)^{-t+1} f([\gamma, t-1]) d\lambda^{\alpha(x)}(\gamma).\end{aligned}$$

- It is clear that the continuity condition is fullfilled. Let us show then that $(\lambda^{[x,t]})_{[x,t]\in X_\alpha}$ is a Haar system. Let γ' be an element of $\mathcal{G}$, let t be a real number and let f be a function in $C_c\left(\mathcal{G}_\alpha^{[r(\gamma'),t]}\right)$. Then

$$\begin{aligned}\lambda_\alpha^{[r(\gamma'),t]}(f) &= \int_{\mathcal{G}^{r(\gamma')}} \rho_\alpha(\gamma)^{-t} f([\gamma, t]) d\lambda^{r(\gamma')}(\gamma) \\ &= \int_{\mathcal{G}^{s(\gamma')}} \rho_\alpha(\gamma' \cdot \gamma)^{-t} f([\gamma' \cdot \gamma, t]) d\lambda^{s(\gamma')}(\gamma) \\ &= \int_{\mathcal{G}^{s(\gamma')}} \rho_\alpha(\gamma)^{-t} f([\gamma' \cdot \gamma, t]) d\lambda^{s(\gamma')}(\gamma) \\ &= \lambda_\alpha^{[s(\gamma'),t]}(f \circ L_{[\gamma',t]}),\end{aligned}$$

  where the third equality holds in view of remark 5.2.

□

5.2. **C^*-algebra of a suspension groupoid.** Let us recall first the construction of the reduced C^*-algebra $C_r^*(\mathcal{G}, \lambda)$ associated to a groupoid $\mathcal{G}$ with base X and Haar system $\lambda = (\lambda^x)_{x\in X}$. Let $\mathcal{L}^2(\mathcal{G}, \lambda)$ be the $C_0(X)$-Hilbert completion of $C_c(\mathcal{G})$ equipped with the $C_0(X)$-valued scalar product

$$\langle \phi, \phi' \rangle(x) = \int_{\mathcal{G}^x} \bar{\phi}(\gamma^{-1}) \phi'(\gamma^{-1}) d\lambda^x(\gamma)$$

for ϕ and ϕ' in $C_c(\mathcal{G})$ and x in X, i.e the completion of $C_c(\mathcal{G})$ with respect to the norm $\|\phi\| = \sup_{x\in X} \langle \phi, \phi \rangle^{1/2}$. The $C_0(X)$-module structure on $C_c(\mathcal{G})$ extends to $\mathcal{L}^2(\mathcal{G}, \lambda)$ and $\langle \bullet, \bullet \rangle$ extends to a $C_0(X)$-valued scalar product on $\mathcal{L}^2(\mathcal{G}, \lambda)$. Recall

that an operator $T : \mathcal{L}^2(\mathcal{G}, \lambda) \to \mathcal{L}^2(\mathcal{G}, \lambda)$ is called adjointable if there exists an operator $T^* : \mathcal{L}^2(\mathcal{G}, \lambda) \to \mathcal{L}^2(\mathcal{G}, \lambda)$, called the adjoint of T such that

$$\langle T^*\phi, \phi' \rangle = \langle \phi, T\phi' \rangle$$

for all ϕ and ϕ' in $\mathcal{L}^2(\mathcal{G}, \lambda)$. Notice that the adjoint, when it exists is unique and that operator that admits an adjoint are automatically $C_0(X)$-linear and continuous. The set of adjointable operators on $\mathcal{L}^2(\mathcal{G}, \lambda)$ is then a C^*-algebra with respect to the operator norm. Then any f in $C_c(\mathcal{G})$ acts as an adjointable operator on $\mathcal{L}^2(\mathcal{G}, \lambda)$ by convolution

$$f \cdot \phi(\gamma) = \int_{\mathcal{G}^{r(\gamma)}} f(\gamma')\phi(\gamma'^{-1}\gamma) d\lambda^{r(\gamma)}(\gamma')$$

where ϕ is in $C_c(\mathcal{G})$, the adjoint of this operator being given by the action of $f^* : \gamma \mapsto \bar{f}(\gamma^{-1})$. The convolution product provides a structure of involutive algebra on $C_c(\mathcal{G})$ and using the action defined above, this algebra can be viewed as a subalgebra of the C^*-algebra of adjointable operators of $\mathcal{L}^2(\mathcal{G}, \lambda)$. The reduced C^*-algebra $C_r^*(\mathcal{G}, \lambda)$ is then the closure of $C_c(\mathcal{G})$ in the C^*-algebra of adjointable operators of $\mathcal{L}^2(\mathcal{G}, \lambda)$. Namely, if we define for x in X the measure on $\mathcal{G}_x$ by $\lambda_x(\phi) = \int_{\mathcal{G}^x} \phi(\gamma^{-1}) d\lambda^x(\gamma)$ for any ϕ in $C_c(\mathcal{G}_x)$, then $\mathcal{L}^2(\mathcal{G}, \lambda)$ is a continuous field of Hilbert spaces over X with fiber $\mathcal{L}^2(\mathcal{G}_x, \lambda_x)$ at x in X. The corresponding $C_0(X)$-structure on $C_r^*(\mathcal{G}, \lambda)$ is then given for h in $C_0(X)$ by the multiplication by $h \circ s$.

Example 5.5. *Let H be a locally compact group acting on a locally compact space X, and consider the semi-direct product groupoid $X \rtimes H$ equipped with a Haar system arising from the Haar measure on H. Then $C_r^*(X \times H, \lambda^\mu)$ is the usual reduced crossed product $C_0(X) \rtimes_r H$.*

Let us denote for any x in X by ν_x the representation of $C_r^*(\mathcal{G}, \lambda)$ on the fiber $\mathcal{L}^2(\mathcal{G}_x, \lambda_x)$. Then for any f in $C_r^*(\mathcal{G}, \lambda)$, we get that $\|f\|_{C_r^*(\mathcal{G},\lambda)} = \sup_{x \in X} \|\nu_x(f)\|$.

Lemma 5.6. *Let $\mathcal{G}$ be a locally compact groupoid with base space X equipped with a Haar system $\lambda = (\lambda^x)_{x \in X}$ and let $\alpha : \mathcal{G} \to \mathcal{G}$ be an automorphism preserving the Haar System λ. Let us define the continuous map $\rho'_\alpha : \mathcal{G} \to \mathbb{R}$; $\gamma \mapsto \rho_\alpha(\gamma^{-1})$. Then there exists a unique automorphism $\tilde{\alpha}$ of the C^*-algebra $C_r^*(\mathcal{G}, \lambda)$ such that for every f in $C_c(\mathcal{G})$ we have $\tilde{\alpha}(f) = (\rho'_\alpha \rho_\alpha)^{1/2} f \circ \alpha^{-1}$.*

Proof. The map $C_c(\mathcal{G}) \to C_c(\mathcal{G})$; $\phi \mapsto \rho_\alpha'^{1/2} \phi \circ \alpha^{-1}$ extends uniquely to a continuous linear and invertible map $W : \mathcal{L}^2(\mathcal{G}, \lambda) \to \mathcal{L}^2(\mathcal{G}, \lambda)$ such that

$$\langle W \cdot \phi, W \cdot \phi \rangle(x) = \langle \phi, \phi \rangle(\alpha^{-1}(x)),$$

for all x in X. Its inverse W^{-1} is defined by $W^{-1}(\phi) = ({\rho'}_\alpha \circ \alpha)^{-1/2} \phi \circ \alpha$ for all ϕ in $C_c(\mathcal{G})$. Let us define

$$\tilde{\alpha} : C_r^*(\mathcal{G}, \lambda) \to C_r^*(\mathcal{G}, \lambda);\ x \mapsto W \cdot x \cdot W^{-1}.$$

Then $W \cdot f \cdot W^{-1} = (\rho'_\alpha \rho_\alpha)^{1/2} f \circ \alpha^{-1}$ for all f in $C_c(\mathcal{G})$. □

Recall that if A is a C^*-algebra and if β is an automorphism of A then the mapping torus of A is the C^*-algebra

$$A_\beta = \{ f \in C([0,1], A) \text{ such that } \beta(f(1)) = f(0) \}.$$

Namely, the mapping torus A_β can be viewed as the algebra of continuous function $h : \mathbb{R} \to A$ such that $h(t) = \beta(h(t+1))$ for all t in $\mathbb{R}$. In this picture, there is an action of $\mathbb{R}$ on A_β by translations defined for t in $\mathbb{R}$ and f in A_β by

$$t \cdot f(s) = f(s-t)$$

for any s in $\mathbb{R}$. Translations then define a strongly continuous action by automorphisms $\widehat{\beta}$ of $\mathbb{R}$ on A_β. By the mapping torus isomorphism, we have a natural Morita equivalence between $A \rtimes_\beta \mathbb{Z}$ and $A \rtimes_{\widehat{\beta}} \mathbb{R}$.

Let α be an automorphism of a groupoid $\mathcal{G}$ preserving a Haar system λ and with density ρ_α. For a function f in $C_c(\mathcal{G}_\alpha)$, we define $\hat{f}$ in $C_c([0,1] \times \mathcal{G}) \subset C([0,1], C_r^*(\mathcal{G}, \lambda))$ by $\hat{f}(t, \gamma) = \rho_\alpha^{-t/2}(\gamma) {\rho'}_\alpha^{-t/2}(\gamma) f([\gamma, t])$. We can check easily that $\hat{f}$ belongs to the mapping torus $C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$.

Proposition 5.7. *Let $\mathcal{G}$ be a locally compact groupoid with base space X equipped with a Haar system $\lambda = (\lambda^x)_{x \in X}$ and let $\alpha : \mathcal{G} \to \mathcal{G}$ be an automorphism preserving the Haar System λ such that $\rho_\alpha \circ \alpha = \rho_\alpha$. Then there is an unique automorphism of C^*-algebras*

$$\Lambda_\alpha : C_r^*(\mathcal{G}_\alpha, \lambda_\alpha) \to C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$$

such that $\Lambda_\alpha(f) = \hat{f}$ for any f in $C_c(\mathcal{G}_\alpha)$.

Proof. Let f be a function of $C_c(\mathcal{G}_\alpha)$. Then

$$\begin{aligned} \|\hat{f}\|_{C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}} &= \sup_{t \in [0,1]} \|\hat{f}(t, \bullet))\|_{C_r^*(\mathcal{G},\lambda)} \\ &= \sup_{t \in [0,1],\, x \in X} \|\nu_x(\hat{f}(t, \bullet))\| \end{aligned}$$

On the other hand,

$$\|f\|_{C_r^*(\mathcal{G}_\alpha,\lambda_\alpha)} = \sup_{t \in [0,1],\, x \in X} \|\nu_{[x,t]}(f)\|,$$

where $\nu_{[x,t]}$ is the representation of $C_r^*(\mathcal{G}_\alpha, \lambda_\alpha)$ on the fiber $\mathcal{L}^2(\mathcal{G}_{\alpha,[x,t]}, \lambda_{\alpha,[x,t]})$ at $[x,t] \in (X \times \mathbb{R})/\mathcal{A}_{\alpha X}$. If we define for t in $[0,1]$ the map $\pi_t : \mathcal{G} \to \mathcal{G}_\alpha : \gamma \mapsto [\gamma, t]$, then

$$C_c(\mathcal{G}_{[x,t]}) \to C_c(\mathcal{G}_x) : \phi \mapsto {\rho'}_\alpha^{-t/2} \phi \circ \pi_t$$

extends to an isometry $W_t : \mathcal{L}^2(\mathcal{G}_{\alpha,[x,t]}, \lambda_{\alpha,[x,t]}) \to \mathcal{L}^2(\mathcal{G}_x, \lambda_x)$ and W_t conjugate $\nu_{[x,t]}(f)$ and $\nu_x(\hat{f}(t, \bullet))$. Thus $\|\hat{f}\|_{C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha}}} = \|f\|_{C_r^*(\mathcal{G}_\alpha,\lambda_\alpha)}$ and

$$C_c(\mathcal{G}_\alpha) \to C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}};\ f \mapsto \hat{f}$$

extends to a monomorphism $\Lambda_\alpha : C_r^*(\mathcal{G}_\alpha, \lambda_\alpha) \to C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$. The set

$$\mathcal{A}_\alpha = \{h \in C_c([0,1] \times \mathcal{G}) \text{ such that } h(1, \alpha(\gamma)) = {\rho'}_\alpha^{t/2} \rho_\alpha^{t/2} h(0, \gamma) \text{ for all } \gamma \in \mathcal{G}\}$$

is dense in $C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$. Let us define for an element h of $\mathcal{A}_\alpha$ the map $\tilde{h} : \mathcal{G} \times \mathbb{R} \to \mathbb{C}$ as the unique map such that

- $\tilde{h}(\gamma, t) = {\rho'}_\alpha^{-t/2} \rho_\alpha^{-t/2} h(t, \gamma)$ for all γ in $\mathcal{G}$ and t in $[0,1]$;
- $h(\alpha(\gamma), t) = h(\gamma, t+1)$ for all γ in $\mathcal{G}$ and t in $\mathbb{R}$.

Then $\tilde{h}$ defines a continuous map of $C_c(\mathcal{G}_\alpha)$ whose image under Λ_α is h. Hence Λ_α has dense range in $C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$ and thus is surjective. □

Remark 5.8. *With the notations of above proposition, let us define for a real s the automorphism of groupoid $\theta_s : \mathcal{G}_\alpha \to \mathcal{G}_\alpha$; $[\gamma, t] \mapsto [\gamma, s+t]$. Then θ_s is preserving the Haar system $\lambda_\alpha = (\lambda^{[x,t]})_{[x,t]\in X_\alpha}$ with density*

$$\mathcal{G}_\alpha \to \mathbb{R};\ [\gamma, t] \mapsto \rho_\alpha(\gamma)^s.$$

We obtain from lemma 5.6 an automorphism $\tilde{\theta}_s$ of $C_r^*(\mathcal{G}_\alpha, \lambda_\alpha)$ which gives rise to a strongly continuous action of $\mathbb{R}$ on $C_r^*(\mathcal{G}_\alpha, \lambda_\alpha)$ by automorphism. The isomorphism

$$\Lambda_\alpha : C_r^*(\mathcal{G}_\alpha, \lambda_\alpha) \to C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$$

of proposition 5.7 is then $\mathbb{R}$-equivariant, where the action of $\mathbb{R}$ on $C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$ is the action $\widehat{\tilde{\alpha}}$ associated to a mapping torus.

6. The dynamic of the uncolored Penrose tiling under translations

As we have seen before, the closure $X_{\mathcal{P}}^{\mathcal{N}}$ of $\mathcal{N}\cdot\mathcal{P}$ for the tiling topology is the suspension $(\Omega\times\mathbb{R})/\mathcal{A}_o$ of the odometer homeomorphism $o : \Omega \to \Omega$; $x \mapsto x+1$, where Ω is the dyadic completion of the integers. The $\mathbb{R}$-algebra $C((\Omega\times\mathbb{R})/\mathcal{A}_o)$ is then the mapping torus algebra of $C(\Omega)$ with respect to automorphism induced by o. In consequence, the crossed product algebras $C(X_{\mathcal{P}}^{\mathcal{N}}) \rtimes \mathbb{R}$ and $C(\Omega)\rtimes\mathbb{Z}$ are Morita equivalent. The purpose of this section is to recall the explicit description of the isomorphism $C(\Omega)\rtimes\mathbb{Z} \overset{\cong}{\to} C(X_{\mathcal{P}}^{\mathcal{N}})\rtimes\mathbb{R}$ arising from this Morita equivalence.

For this, let us define on $C_c(\Omega\times\mathbb{R})$ the $C(\Omega)\rtimes\mathbb{Z}$-valued inner product

$$\langle \xi,\, \xi'\rangle(\omega, k) = \int_{\mathbb{R}} \bar{\xi}(\omega, s)\xi'(\omega - k, s+k)ds$$

for ξ and ξ' in $C_c(\Omega\times\mathbb{R})$ and (ω, k) in $(\Omega\times\mathbb{R})$. This inner product is positive and gives rise to a right $C(\Omega)\rtimes\mathbb{Z}$-Hilbert module $\mathcal{E}$, the action of $C(\Omega)\rtimes\mathbb{Z}$ being given for h in $C_c(\Omega\times\mathbb{Z})$ and ξ in $C_c(\Omega\times\mathbb{R})$ by

$$\xi\cdot h(\omega, t) = \sum_{n\in\mathbb{Z}} \xi(n+\omega, t-n)h(n+\omega, n)$$

for (ω, k) in $\Omega\times\mathbb{R}$. The right $C(\Omega)\rtimes\mathbb{Z}$-Hilbert module $\mathcal{E}$ is also equipped with a left action of $C\left((\Omega\times\mathbb{R})/\mathcal{A}_o\right)\rtimes\mathbb{R}$ given for f in $C_c\left((\Omega\times\mathbb{R})/\mathcal{A}_o\times\mathbb{R}\right)$ and ξ in $C_c(\Omega\times\mathbb{R})$ by

$$f\cdot\xi(\omega, t) = \int_{\mathbb{R}} f([\omega, t], s)\xi(\omega, t-s)ds$$

for (ω, k) in $\Omega\times\mathbb{R}$. We get in this way a $C\left((\Omega\times\mathbb{R})/\mathcal{A}_o\right)\rtimes\mathbb{R} - C(\Omega)\rtimes\mathbb{Z}$ imprimitivity bimodule which implements the Morita equivalence we are looking for. Actually, there is an isomorphism of right $C(\Omega)\rtimes\mathbb{Z}$-Hilbert module

$$\Psi : \mathcal{E} \to L^2([0,1])\otimes C(\Omega)\rtimes\mathbb{Z}$$

defined in a unique way by $\Psi(g) = g\otimes u$ for g in $C_c(\mathbb{R})$ supported in $(0,1)$, where u is the unitary of $C(\Omega)\rtimes\mathbb{Z}$ corresponding to the positive generator of $\mathbb{Z}$. Using the right $C\left((\Omega\times\mathbb{R})/\mathcal{A}_o\right)\rtimes\mathbb{R}$-module structure of the $C\left((\Omega\times\mathbb{R})/\mathcal{A}_o\right)\rtimes\mathbb{R} - C(\Omega)\rtimes\mathbb{Z}$ imprimitivity bimodule $\mathcal{E}$ and the isomorphism Ψ, we get an isomorphism

$$C\left((\Omega\times\mathbb{R})/\mathcal{A}_o\right)\rtimes\mathbb{R} \overset{\cong}{\to} \mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}. \tag{6.1}$$

This isomorphism can be described as follows. Let us define for f and g in $L^2([0,1])$ the rank one operator

$$\Theta_{f,g} : L^2([0,1]) \to L^2([0,1]);\ h \mapsto f\langle g, h\rangle.$$

We define for ξ and ξ' in $C_c(\Omega\times\mathbb{R})$ the continuous function of $C_c\left((\Omega\times\mathbb{R})/\mathcal{A}_o\times\mathbb{R}\right)$

$$\Theta^{\Omega}_{\xi,\xi'}([\omega,s],t)=\sum_{k\in\mathbb{Z}}\xi(\omega+k,s-k)\bar{\xi}'(\omega+k,s-t-k)$$

for all ω in Ω and s and t in $\mathbb{R}$. It is straightforward to check that $\Theta^{\Omega}_{\xi,\xi'}$ is well defined and that

$$\Theta^{\Omega}_{\xi,\xi'}\cdot\eta=\xi\langle\xi',\eta\rangle$$

for all η in $C_c(\Omega\times\mathbb{R})$. If we set for f and g in $C_c(\mathbb{R})$ with support in $(0,1)$ and for ϕ in $C(\Omega)$, $\xi=1\otimes f$, $\xi'=\phi\otimes g$ and $\xi'':\Omega\times\mathbb{R}\to\mathbb{R};(\omega,t)\mapsto g(t+1)$, then the image of $\Theta^{\Omega}_{\xi,\xi'}$ under the isomorphism of equation (6.1) is $\Theta_{f,g}\otimes\phi\in\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$ and moreover,

$$\Theta^{\Omega}_{\xi,\xi'}([\omega,s],t)=\sum_{k\in\mathbb{Z}}f(s-k)\bar{\phi}(\omega+k)\bar{g}(s-t-k). \tag{6.2}$$

The image of $\Theta^{\Omega}_{\xi,\xi''}$ under the isomorphism of equation (6.1) is $\Theta_{f,g}\otimes u\in\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$ and moreover,

$$\Theta^{\Omega}_{\xi,\xi''}([\omega,s],t)=\sum_{k\in\mathbb{Z}}f(s-k)\bar{g}(s+1-t-k). \tag{6.3}$$

Let us define the automorphism α of the groupoid $(\Omega\times\mathbb{R})/\mathcal{A}_o\rtimes\mathbb{R}$ in the following way

- $\alpha([\omega,s],t)=([\omega/2,s/2],t/2)$ if ω is even;
- $\alpha([\omega,s],t)=([(\omega+1)/2,(s+1)/2],t/2)$ if ω is odd.

Notice that $\alpha^{-1}([\omega,s],t)=([2\omega,2s],2t)$ for all ω in Ω and s and t in $\mathbb{R}$. Then α preserves the Haar system of $(\Omega\times\mathbb{R})/\mathcal{A}_o\rtimes\mathbb{R}$ arising from the Haar mesure on $\mathbb{R}$ and has constant density $\rho_\alpha=2$. Hence according to lemma 5.6, the automorphism of groupoid α induces an automorphism $\tilde{\alpha}$ of C^*-algebra $C\left((\Omega\times\mathbb{R})/\mathcal{A}_o\right)\rtimes\mathbb{R}$ such that $\tilde{\alpha}(h)=2h\circ\alpha^{-1}$ for all h in $C\left((\Omega\times\mathbb{R})/\mathcal{A}_o\times\mathbb{R}\right)$. We are now in position to describe how $\tilde{\alpha}$ is transported under the isomorphism of equation (6.1) to an automorphism Υ of $\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$. With ξ, ξ' and ξ'' as defined above,

$$\begin{aligned}
\tilde{\alpha}(\Theta^{\Omega}_{\xi,\xi'})([\omega,s],t) &= 2\Theta^{\Omega}_{\xi,\xi'}([2\omega,2s],2t)\\
&= 2\sum_{k\in\mathbb{Z}}f(2s-k)\bar{\phi}(2\omega+k)\bar{g}(2s-2t-k)\\
&= 2\sum_{k\in\mathbb{Z}}f(2s-2k)\bar{\phi}(2\omega+2k)\bar{g}(2s-2t-2k)+ \qquad (6.4)\\
&\quad 2\sum_{k\in\mathbb{Z}}f(2s-2k-1)\bar{\phi}(2\omega+2k-1)\bar{g}(2s-2t-2k-1)
\end{aligned}$$

and

$$\begin{aligned}\tilde{\alpha}(\Theta^{\Omega}_{\xi,\xi''})([\omega,s],t) &= 2\Theta^{\Omega}_{\xi,\xi''}([2\omega,2s],2t)\\ &= 2\sum_{k\in\mathbb{Z}} f(2s-k)\bar{g}(2s+1-2t-k)\\ &= 2\sum_{k\in\mathbb{Z}} f(2s-2k)\bar{g}(2s+1-2t-2k)+ \\ &\qquad 2\sum_{k\in\mathbb{Z}} f(2s-2k-1)\bar{g}(2s-2t-2k). \end{aligned} \tag{6.5}$$

To complete the description of the automorphism Υ of $\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$ corresponding to $\tilde{\alpha}$, we need to introduce some further notations. We define the partial isometries U_0, U_1 and V of $L^2([0,1])$ by

- $U_0 f(t) = \sqrt{2}f(2t)$ if $t\in[0,1/2]$ and $U_0 f(t)=0$ otherwise;
- $U_1 f(t) = \sqrt{2}f(2t-1)$ if $t\in[1/2,1]$ and $U_1 f(t)=0$ otherwise;
- $Vf(t) = f(t+1/2)$ if $t\in[0,1/2]$ and $Vf(t)=0$ otherwise,

for f in $C([0,1])$. Let use define also the endomorphisms W_0 and W_1 of the C^*-algebra $C(\Omega)$ by $W_0\phi(\omega)=\phi(2\omega)$ and $W_1\phi(\omega)=\phi(2\omega+1)$, for ϕ in $C(\Omega)$ and ω in Ω. Using this notations, equations (6.4) and (6.5) can be rewriten as

$$\tilde{\alpha}(\Theta^{\Omega}_{\xi,\xi'})([\omega,s],t) = \sum_{k\in\mathbb{Z}} U_0 f(s-k)W_0\bar{\phi}(\omega+k)U_0\bar{g}(s-t-k)+\sum_{k\in\mathbb{Z}} U_1 f(s-k)W_1\bar{\phi}(\omega+k)U_1\bar{g}(s-t-k)$$

and

$$\tilde{\alpha}(\Theta^{\Omega}_{\xi,\xi''})([\omega,s],t) = \sum_{k\in\mathbb{Z}} U_0 f(s-k)U_1\bar{g}(s-t-k+1)+\sum_{k\in\mathbb{Z}} U_1 f(s-k)U_0\bar{g}(s-t-k).$$

Thus, in view of equations (6.2) and (6.3), we get that

$$\Upsilon(\Theta_{f,g}\otimes\phi) = \Theta_{U_0 f,U_0 g}\otimes W_0\phi + \Theta_{U_1 f,U_1 g}\otimes W_1\phi$$

and

$$\Upsilon(\Theta_{f,g}\otimes u) = \Theta_{U_0 f,U_1 g}\otimes u + \Theta_{U_1 f,U_0 g}\otimes 1.$$

From this we deduce

$$\Upsilon(k\otimes\phi) = U_0\cdot k\cdot U_0^*\otimes W_0\phi + U_1\cdot k\cdot U_1^*\otimes W_1\phi$$

and

$$\begin{aligned}\Upsilon(k\otimes u) &= U_0\cdot k\cdot U_1^*\otimes u + U_1\cdot k\cdot U_0^*\otimes 1\\ &= U_0\cdot k\cdot U_0^*\cdot V\otimes u + U_1\cdot k\cdot U_1^*\cdot V^*\otimes 1\\ &= (U_0\cdot k\cdot U_0^* + U_1\cdot k\cdot U_1^*)\cdot(V\otimes u + \cdot V^*\otimes 1)\end{aligned}$$

where the second equality holds since $V^*\cdot U_0 = U_1$ and $V\cdot U_1 = U_0$ and the third holds since $V^*U_1 = VU_0 = 0$. In consequence, if we extends Υ to the multiplier algebra of $\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$, we finally obtain that the automorphism Υ is the unique homomorphism of C^*-algebra such that

$$\Upsilon(k\otimes\phi) = U_0^*\cdot k\cdot U_0\otimes W_0\phi + U_1^*\cdot k\cdot U_1\otimes W_1\phi$$

and

$$\Upsilon(1\otimes u) = V\otimes u + V^*\otimes 1, \tag{6.6}$$

where k is in $\mathcal{K}(L^2([0,1]))$, ϕ is in $C(\Omega)$ and $1\otimes u$ and $V\otimes u + V^*\otimes 1$ are viewed as multipliers of $\mathcal{K}(L^2([0,1]))\otimes C(\Omega)\rtimes\mathbb{Z}$.

The following lemma will be helpful to compute the K-theory of the C^*-algebra of the Penrose hyperbolic tiling. For short, we will denote from now on $\mathcal{K}(L^2([0,1]))$ by $\mathcal{K}$.

Lemma 6.1. *Let A be the unitarisation of $\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}$ and let f be a norm one function of $L^2([0,1])$. Then the unitaries*

$$(1-\Theta_{f,f}\otimes 1)+\Theta_{f,f}\otimes u$$

and

$$\Theta_{U_0f,U_1f}\otimes u+\Theta_{U_1f,U_0f}\otimes 1+1-\Theta_{U_0f,U_0f}\otimes 1-\Theta_{U_1f,U_1f}\otimes 1 \tag{6.7}$$

of A are homotopic.

Proof. If we set $f_0=f$ and complete to a Hilbertian base $f_0,\dots,f_n,\dots$ of $L^2([0,1])$, then $U_0f_0,\dots,U_0f_n,\dots;U_1f_0,\dots,U_1f_n,\dots$ is a Hilbertian basis of $L^2([0,1])$. In this base the unitary of equation (6.7) can be written down as

$$\left(\begin{array}{ccc|ccc} 0 & & & u & & \\ & 1 & & & 0 & \\ & & \ddots & & & \ddots \\ \hline 1 & & & 0 & & \\ & 0 & & & 1 & \\ & & \ddots & & & \ddots \end{array}\right)$$

which is homotopic to

$$\left(\begin{array}{ccc|ccc} u & & & 0 & & \\ & 1 & & & 0 & \\ & & \ddots & & & \ddots \\ \hline 0 & & & 1 & & \\ & 0 & & & 1 & \\ & & \ddots & & & \ddots \end{array}\right).$$

All unitaries that can be writen down in such way in some hilbertian basis of $L^2([0,1])$ are homotopic and since this is the case for $1-\Theta_{f,f}\otimes 1+\Theta_{f,f}\otimes u$, we get the result. □

7. The C^*-algebra of a Penrose hyperbolic tiling

Let us consider the semi-direct product groupoid $\mathcal{G}=(X_{\mathcal{P}}^{\mathcal{N}}\times Z_w)\rtimes\mathbb{R}$ corresponding to the diagonal action of $\mathbb{R}$ on $X_{\mathcal{P}}^{\mathcal{N}}\times Z_w$, by translations on $X_{\mathcal{P}}^{\mathcal{N}}$ and trivial on Z_ω. Let us denote by $\lambda=(\lambda_{(\mathcal{P}',\omega)})_{(\mathcal{P}',\omega)\in X_{\mathcal{P}}^{\mathcal{N}}\times Z_w}$ the Haar system provided by the left Haar mesure on $\mathbb{R}$. Let us define the groupoid automorphism $\alpha_w:\mathcal{G}\to\mathcal{G}$; $(\mathcal{P}',w',t)\mapsto(R\cdot\mathcal{P}',\sigma(w'),2t)$. Then α_w preserves the Haar system λ with constant density $\rho_{\alpha_w}=1/2$ and thus according to lemma 5.4 the suspension groupoid $\mathcal{G}_{\alpha_w}$ admits a Haar system λ_{α_w}. The semi-direct product groupoid $X_{\mathcal{P}(w)}^G\rtimes\mathbb{R}$, where $\mathbb{R}$ acts on $X_{\mathcal{P}(w)}^G$ by translations, is equipped with an action of $\mathbb{R}$ by automorphisms $\beta_t:X_{\mathcal{P}(w)}^G\rtimes\mathbb{R}\to X_{\mathcal{P}(w)}^G\rtimes\mathbb{R}$; $(\mathcal{T},s)\mapsto(2^t\cdot\mathcal{T},2^ts)$ for any t in $\mathbb{R}$. The automorphism β_t preserves the Haar system with constant density 2^{-t} and thus in view of proposition 5.7 induced a strongly continuous action of $\mathbb{R}$ on the crossed product C^*-algebra $C(X_{\mathcal{P}(w)}^G)\rtimes\mathbb{R}$.

Lemma 7.1. *Let w be an element of $\{1,\dots,r\}^{\mathbb{Z}}$. Then there is a unique isomorphism of groupoids $\Phi_w : \mathcal{G}_{\alpha_w} \longrightarrow X^G_{\mathcal{P}(w)} \rtimes \mathbb{R}$ such that:*

1. *$\Phi_w([\mathcal{P}+x, w, y, 0]) = (\mathcal{P}(w)+x, y)$ for all x and y in $\mathbb{R}$;*
2. *Φ_w is equivariant with respect to the actions of $\mathbb{R}$;*
3. *$\Phi_{w,*}\lambda_{\alpha_w}$ is the Haar system on $X^G_{\mathcal{P}(w)} \rtimes \mathbb{R}$ provided by the Haar measure on $\mathbb{R}$.*

Proof. With notations of the proof of proposition 4.3, let us define $\mathcal{T}(w') = \overline{\Phi}([T, w, 1])$, were $\mathcal{T}$ is in $X^{\mathcal{N}}_{\mathcal{P}}$ and w' is in $\{1, ..., r\}^{\mathbb{Z}}$. Then the map

$$X^{\mathcal{N}}_{\mathcal{P}} \times Z_w \to X^G_{\mathcal{P}(w)};\ (\mathcal{T}, w') \mapsto \mathcal{T}(w')$$

is continuous and since $(R\cdot\mathcal{T})(\sigma(w')) = R\cdot\mathcal{T}(w')$, the continuous map

$$\mathcal{G}\times\mathbb{R} \to X^G_{\mathcal{P}(w)} \rtimes \mathbb{R};\ (\mathcal{T}, w', x, y) \mapsto (R_{2^y}\mathcal{T}(w'), 2^y x)$$

induces a continuous homomorphism of groupoids

$$\Phi_w : \mathcal{G}_{\alpha_w} \to X^G_{\mathcal{P}(w)} \rtimes \mathbb{R}.$$

This map is clearly one-to-one since the equality $R_{2^t}\mathcal{T}(w') = \mathcal{T}'(w'')$ for t in $\mathbb{R}$, $\mathcal{T}$ and $\mathcal{T}'$ in $X^{\mathcal{N}}_{\mathcal{P}}$ and w'' and w' in Z_w holds if and only if t is integer, $w'' = \sigma^t(w')$ and $R_{2^t}\mathcal{T} = \mathcal{T}'$. To prove that Φ_w is onto, let us remark that any element of $X^G_{\mathcal{P}(w)}$ can be written as $R_{2^a}\mathcal{T}(w')$, with a in $\mathbb{R}$, $\mathcal{T}$ in $X^{\mathcal{N}}_{\mathcal{P}}$ and w' in Z_ω. We get then

$$\Phi_w([\mathcal{T}, w', 2^{-a}t, a]) = (R_{2^a}\mathcal{T}(w'), t)$$

for all t in $\mathbb{R}$.

It is then straightforward to check that condition (3) of the lemma is satisfied. The uniqueness of Φ_w is a consequence on one hand of its equivariance and on the other hand of the density of the $\mathbb{R}$-orbit of $\mathcal{P}$ in $X^{\mathcal{N}}_{\mathcal{P}}$. □

As a consequence of lemma 7.1, we get

Corollary 7.2. *The map*

$$C_c(X^G_{\mathcal{P}(w)} \rtimes \mathbb{R}) \to C_c(\mathcal{G}_{\alpha_w});\ f \mapsto f\circ\Phi_w$$

induces an $\mathbb{R}$-equivariant isomorphism

$$\widetilde{\Phi}_w : C_0(X^G_{\mathcal{P}(w)}) \rtimes \mathbb{R} \to C^*_r(\mathcal{G}_{\alpha_w}, \lambda_{\alpha_w}).$$

Proposition 7.3. *Using the notations of lemmas 5.6 and 7.1, the C^*-algebras $C(X^G_{\mathcal{P}(w)}) \rtimes G$ and $C^*_r(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_w}} \mathbb{Z}$ are Morita equivalent.*

Proof. Recall that $G = \mathbb{R} \rtimes \mathbb{R}^*_+$, where the group $(\mathbb{R}^*_+, \cdot)$ acts on $(\mathbb{R}, +)$ by multiplication. Iterate crossed products leads to an isomorphism

$$C(X^G_{\mathcal{P}(w)}) \rtimes G \cong (C(X^G_{\mathcal{P}(w)}) \rtimes \mathbb{R}) \rtimes \mathbb{R}^*_+.$$

If we identify the groups $(\mathbb{R}, +)$ and $(\mathbb{R}^*_+, \cdot)$ using the isomorphism

$$\mathbb{R} \to \mathbb{R}^*_+;\ t \mapsto 2^t,$$

this provides the action under consideration in lemma 7.1 of $\mathbb{R}$ on $C(X^G_{\mathcal{P}(w)}) \rtimes \mathbb{R}$ and hence, the algebras $C(X^G_{\mathcal{P}(w)}) \rtimes G$ and $C^*(\mathcal{G}_{\alpha_w}, \lambda_{\alpha_w}) \rtimes \mathbb{R}$ are isomorphic. In view of lemma 5.7 and of remark 5.8, the C^*-algebra $C(X^G_{\mathcal{P}(w)}) \rtimes G$ is isomorphic to $C^*_r(\mathcal{G},\lambda)_{\tilde{\alpha_w}} \rtimes \mathbb{R}$. But since $C^*_r(\mathcal{G},\lambda)_{\tilde{\alpha_w}}$ is the mapping torus algebra with respect to the automorphism $\tilde{\alpha_w} : C^*_r(\mathcal{G},\lambda) \to C^*_r(\mathcal{G},\lambda)$, the crossed product C^*-algebra

$C_r^*(\mathcal{G},\lambda)_{\tilde{\alpha_w}} \rtimes \mathbb{R}$ is Morita equivalent to $C_r^*(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_w}} \mathbb{Z}$ and hence we get the result. □

8. The K-theory of the C^*-algebra of a Penrose hyperbolic tiling

Let us consider the semi-direct groupoid $\mathcal{G} = (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w) \rtimes \mathbb{R}$ corresponding to the diagonal action of $\mathbb{R}$ on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w$, by translations on $X_{\mathcal{P}}^{\mathcal{N}}$ and trivial on Z_w. According to proposition 7.3 we have an isomorphism

$$K_*(C(X_{\mathcal{P}(w)}^G) \rtimes G) \stackrel{\cong}{\to} K_*(C_r^*(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_w}} \mathbb{Z})$$

induced by the Morita equivalence. In order to compute this K-theory group, we need to recall some basic facts concerning the K-theory group of a crossed product of a C^*-algebra A by an action of $\mathbb{Z}$ provided by an automorphism θ of A. This K-theory can be computed by using the Pimsner-Voiculescu exact sequence [18]

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\theta_*-Id} & K_0(A) & \xrightarrow{\iota_*} & K_0(A \rtimes_\theta \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\theta \mathbb{Z}) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{\theta_*-Id} & K_1(A) \end{array},$$

where ι_* is the homomorphism induced in K-theory by the inclusion $\iota : A \hookrightarrow A \rtimes_\theta \mathbb{Z}$ and θ_* is the homomorphism in K-theory induced by θ. The vertical maps are given by the composition

$$K_*(A \rtimes_\theta \mathbb{Z}) \xrightarrow{\cong} K_*(A_\theta \rtimes_{\widehat{\theta}} \mathbb{R}) \xrightarrow{\cong} K_{*+1}(A_\theta) \xrightarrow{ev_*} K_{*+1}(A),$$

where

- A_θ is the mapping torus of A with respect to the action θ endowed, with its associated action $\widehat{\theta}$ of $\mathbb{R}$;
- the first map is induced by the Morita equivalence between $A \rtimes_\theta \mathbb{Z}$ and $A_\theta \rtimes_{\widehat{\theta}} \mathbb{R}$;
- the second map is the Thom-Connes isomorphism;
- the third map is induced in K-theory by the evaluation map

$$ev : A_\theta \to A;\ f \mapsto f(0).$$

For an automorphism Ψ of an abelian group M, let us define $\operatorname{Inv} M$ as the set of invariant elements of M and by $\operatorname{Coinv} M = M/\{x - \Psi(x), x \in M\}$ the set of coinvariant elements. We then get short exact sequences

$$0 \to \operatorname{Coinv} K_0(A) \to K_0(A \rtimes_\theta \mathbb{Z}) \to \operatorname{Inv} K_1(A) \to 0 \tag{8.1}$$

and

$$0 \to \operatorname{Coinv} K_1(A) \to K_1(A \rtimes_\theta \mathbb{Z}) \to \operatorname{Inv} K_0(A) \to 0. \tag{8.2}$$

Moreover the inclusions in these exact sequences are induced by ι_*. The first step in the computation of $K_*(C_r^*(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_w}} \mathbb{Z})$ is provided by next lemma, which is straightforward to prove.

Lemma 8.1. *Let Z be a Cantor set and let us denote by $C(Z,\mathbb{Z})$ the algebra of continuous and integer valued functions on Z.*

1. *we have an isomorphism $C(Z,\mathbb{Z}) \to K_0(C(Z))$; $\chi_E \mapsto [\chi_E]$.*
2. *$K_1(C(Z)) = \{0\}$,*

where for a compact-open subset E of Z, then χ_E stands for the characteristic function of E.

Plugging $C_r^*(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_w}} \mathbb{Z}$ into the short exact sequences (8.1) and (8.2), we get

$$0 \to \operatorname{Coinv} K_0(C_r^*(\mathcal{G},\lambda)) \to K_0(C_r^*(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_w}} \mathbb{Z}) \to \operatorname{Inv} K_1(C_r^*(\mathcal{G},\lambda)) \to 0 \tag{8.3}$$

and

$$0 \to \operatorname{Coinv} K_1(C_r^*(\mathcal{G},\lambda)) \to K_1(C_r^*(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_w}} \mathbb{Z}) \to \operatorname{Inv} K_0(C_r^*(\mathcal{G},\lambda)) \to 0. \tag{8.4}$$

According to equation (6.1), the C^*-algebra $C_r^*(\mathcal{G},\lambda)$ is isomorphic to $C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}$. The K-theory of $C_r^*(\mathcal{G},\lambda)$ can be the computed by using the Künneth formula: in view of lemma 8.1, $K_0(C(Z_w)) \cong C(Z_w,\mathbb{Z})$ is torsion free and $K_1(C(Z_w)) = \{0\}$ and by Morita equivalence, we get that

$$K_0(C_r^*(\mathcal{G},\lambda)) \cong C(Z_w,\mathbb{Z}) \otimes K_0(C(\Omega) \rtimes \mathbb{Z})$$

and

$$K_1(C_r^*(\mathcal{G},\lambda)) \cong C(Z_w,\mathbb{Z}) \otimes K_1(C(\Omega) \rtimes \mathbb{Z}).$$

This isomorphism, up to the Morita equivalence and to the isomorphism of equation (6.1) are implemented by the external product in K-theory and will be precisely described later on. Once again, $K_*(C(\Omega) \rtimes \mathbb{Z})$ can be computed from the short exact sequences (8.1) and (8.2), and we get, using lemma 8.1 that

$$K_0(C(\Omega) \rtimes \mathbb{Z}) \cong \operatorname{Coinv} C(\Omega,\mathbb{Z}) \tag{8.5}$$

and

$$K_1(C(\Omega) \rtimes \mathbb{Z}) \cong \operatorname{Inv} C(\Omega,\mathbb{Z}) \cong \mathbb{Z}. \tag{8.6}$$

The isomorphism of equation (8.5) is induced by the composition

$$C(\Omega,Z) \xrightarrow{\cong} K_0(C(\Omega)) \to K_0(C(\Omega) \rtimes \mathbb{Z}),$$

which factorizes through $\operatorname{Coinv} C(\Omega,Z)$, where the first map is described in lemma 8.1, and the second map is induced on K-theory by the inclusion $C(\Omega) \hookrightarrow C(\Omega) \rtimes \mathbb{Z}$. In the first isomorphism of equation (8.6) the class of $[u]$ in $K_1(C(\Omega) \rtimes \mathbb{Z})$ of the unitary u of $C(\Omega) \rtimes \mathbb{Z}$ corresponding to the positive generator of $\mathbb{Z}$ is mapped to the constant function 1 of $C(\Omega,\mathbb{Z})$.

Lemma 8.2. *Let ν be the Haar measure on Ω. Then*

1. *$\int f d\nu$ is in $\mathbb{Z}[1/2]$ for all f in $C(\Omega,\mathbb{Z})$;*
2. *$C(\Omega,\mathbb{Z}) \to \mathbb{Z}[1/2]; f \mapsto \int f d\nu$ factorizes through an isomorphism*

$$\operatorname{Coinv} C(\Omega,\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}[1/2].$$

Proof. It is enought to check the first point for characteristic function of compact-open subset of Ω. For an integer n and k in $\{0,\dots,2^{n-1}\}$, we set $F_{n,k} = 2^n\Omega + k$. Then $(F_{n,k})_{n\in\mathbb{N},\,0\le k\le 2^{n-1}}$ is a basis of compact-open neighborhoods for Ω and thereby, every compact-open subset of Ω is a finite disjoint union of some $F_{n,k}$. Since $\nu(F_{n,k}) = 2^{-n}$, we get the first point.

The measure μ being invariant by translation, the map

$$C(\Omega,\mathbb{Z}) \to \mathbb{Z}[1/2]; f \mapsto \int f d\nu$$

factorizes through a group homomorphism $\mathrm{Coinv}\, C(\Omega,\mathbb{Z}) \to \mathbb{Z}[1/2]$. This homomorphism admits a cross-section

$$\mathbb{Z}[1/2] \to \mathrm{Coinv}\, C(\Omega,\mathbb{Z});\ 2^{-n} \mapsto [\chi_{F_{n,0}}]. \tag{8.7}$$

This map is well defined since $F_{n,0} = F_{n+1,0} \coprod (2^n + F_{n+1,0})$ and thus

$$[\chi_{F_{n,0}}] = [\chi_{F_{n+1,0}}] + [\chi_{2^n+F_{n+1,0}}] = 2[\chi_{F_{n+1,0}}]$$

in $\mathrm{Coinv}\, C(\Omega,\mathbb{Z})$. Since the $(\chi_{F_{n,k}})_{n\in\mathbb{N},\, 0\leq k\leq 2^{n-1}}$ generates $C(\Omega,\mathbb{Z})$ as an abelian group, it is enought to check that the cross-section of equation (8.7) is a left inverse on $\chi_{F_{n,k}}$, which is true since $[\chi_{F_{n,k}}] = [\chi_{k+F_{n,0}}] = [\chi_{F_{n,0}}]$ in $\mathrm{Coinv}\, C(\Omega,\mathbb{Z})$. □

Proposition 8.3. *Let $C(Z_w,\mathbb{Z}[1/2]) \cong C(Z_w,\mathbb{Z}) \otimes \mathbb{Z}[1/2]$ be the algebra of continuous function on Z_w, valued in $\mathbb{Z}[1/2]$ (equipped with the discrete topology). Then with the notations of the proof of lemma 8.2, we have isomorphisms*

(1)

$$\begin{array}{rcl} C(Z_w,\mathbb{Z}[1/2]) & \xrightarrow{\cong} & K_0(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z}) \\ \dfrac{\chi_E}{2^n} & \mapsto & [\chi_E \otimes \chi_{F_{n,0}}], \end{array}$$

where E is a compact-open subset of Z_w and χ_E is its characteristic function.

(2)

$$\begin{array}{rcl} C(Z_w,\mathbb{Z}) & \xrightarrow{\cong} & K_1(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z}) \\ \chi_E & \mapsto & [\chi_E \otimes u + (1-\chi_E)\otimes 1], \end{array}$$

where u is the unitary of $C(\Omega)\rtimes\mathbb{Z}$ corresponding to the positive generator of $\mathbb{Z}$.

Proof. As we have already mentionned, $K_*(C(Z_w))$ is torsion free and the Künneth formula provides isomorphisms

$$\begin{array}{rcl} K_0(C(Z_w)) \otimes K_0(C(\Omega)\rtimes\mathbb{Z}) & \xrightarrow{\cong} & K_0(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z}) \\ [p]\otimes[q] & \mapsto & [p\otimes q], \end{array}$$

where p and q are some matrix projectors with coefficients respectively in $C(Z_w)$ and $C(\Omega)\rtimes\mathbb{Z}$, and

$$\begin{array}{rcl} K_0(C(Z_w)) \otimes K_1(C(\Omega)\rtimes\mathbb{Z}) & \xrightarrow{\cong} & K_1(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z}) \\ [p]\otimes[v] & \mapsto & [p\otimes v + (I_k - p)\otimes I_l], \end{array}$$

where p is a projector in $M_l(C(Z_w))$ and v is a unitary in $M_k(C(\Omega)\rtimes\mathbb{Z})$. The proposition is then a consequence of lemmas 8.1, 8.2 and of the discussion related to equations (8.5) and (8.6). □

In order to compute the invariants and the coinvariants of

$$K_*(C_r^*(\mathcal{G},\lambda)) \cong K_*(C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}),$$

we will need a carefull description of the action induced in K-theory by the automorphism $\sigma^*\otimes\Upsilon$ of $C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z}$, where σ^* is the automorphism of $C(Z_\omega)$ induced by the shift σ and where Υ was defined in section 6.

Lemma 8.4. *If we equip $C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}$ with the $\mathbb{Z}$-action provided by $\sigma^* \otimes \Upsilon$ and under the $\mathbb{Z}$-equivariant isomorphism*

$$C_r^*(\mathcal{G}, \lambda) \cong C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z},$$

the action induced by α_w on $K_0(C_r^*(\mathcal{G}, \lambda) \cong C(Z_w, \mathbb{Z}[1/2])$ and on $K_1(C_r^*(\mathcal{G}, \lambda) \cong C(Z_w, \mathbb{Z})$ are given by the automorphisms of abelian groups

$$\begin{array}{rcl} \Psi_0 : C(Z_w, \mathbb{Z}[1/2]) & \to & C(Z_w, \mathbb{Z}[1/2]) \\ f & \mapsto & 2f \circ \sigma^{-1} \end{array}$$

and

$$\begin{array}{rcl} \Psi_1 : C(Z_w, \mathbb{Z}) & \to & C(Z_w, \mathbb{Z}) \\ f & \mapsto & f \circ \sigma^{-1}. \end{array}$$

Proof. According to proposition 8.3 and using the Morita equivalence between $C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z}$ and $C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}$, in order to describe Ψ_0, we have to compute the image under $(\sigma^* \otimes \Upsilon)_*$ of

$$[\chi_E \otimes \Theta_{f,f} \otimes \chi_{F_{n,0}}] \in K_0(C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z})$$

where,

- χ_E is the characteristic function of a compact-open subset E of Z_w;
- $\chi_{F_{n,0}}$ is the characteristic function of $F_{n,0} = 2^n\Omega$ for $n \geq 1$;
- $\Theta_{f,f}$ is the rank one projector associated to a norm 1 function f of $L^2([0,1])$.

We have

$$\begin{array}{rcl} \sigma^* \otimes \Upsilon(\chi_E \otimes \Theta_{f,f} \otimes \chi_{F_{n,0}}) & = & \chi_{\sigma(E)} \otimes \Theta_{U_0 f, U_0 f} \otimes W_0 \chi_{F_{n,0}} + \chi_{\sigma(E)} \otimes \Theta_{U_1 f, U_1 f} \otimes W_1 \chi_{F_{n,0}} \\ & = & \chi_{\sigma(E)} \otimes \Theta_{U_0 f, U_0 f} \otimes \chi_{F_{n-1,0}}, \end{array}$$

where the last equality holds since $W_0 \chi_{F_{n,0}} = \chi_{F_{n-1,0}}$ and $W_1 \chi_{F_{n,0}} = 0$. Since $\Theta_{U_0 f, U_0 f}$ is again a rank one projector, then up to the Morita equivalence between $C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z}$ and $C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}$, the image of $[\chi_E \otimes \chi_{F_{n,0}}] \in K_0(C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z})$ under $(\sigma^* \otimes \Upsilon)_*$ is $[\chi_{\sigma(E)} \otimes \chi_{F_{n-1,0}}] \in K_0(C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z})$. Using proposition 8.3, this completes the description of Ψ_0. For Ψ_1, notice first that up to the isomorphism

$$K_0(C(Z_w)) \otimes K_0(\mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}) \stackrel{\cong}{\to} K_1(C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z})$$

provided by the Künneth formula, the action of $(\sigma^* \otimes \Upsilon)_*$ is $\sigma_*^* \otimes \Upsilon_*$ and then the result is a consequence of lemma 6.1 and of proposition 8.3. □

Let us equip $C(Z_w, \mathbb{Z}[1/2])$ and $C(Z_w, \mathbb{Z})$ with the $\mathbb{Z}$-actions respectively provided by Ψ_0 and Ψ_1. Then since $\|\Psi_0(h)\| = 2\|h\|$ for any h in $C(Z_w, \mathbb{Z}[1/2])$, we get that $\operatorname{Inv} C(Z_w, \mathbb{Z}[1/2]) = \{0\}$ We are now in position to get a complete description of the K-theory of $C(X_{\mathcal{P}(w)}^G) \rtimes G$. In view of the short exact sequences of equations (8.3) and (8.4), the two following theorems are then consequences of lemma 8.4 and of proposition 8.3.

Theorem 8.5. *We have a short exact sequence*

$$0 \to \operatorname{Coinv} C(Z_w, \mathbb{Z}[1/2]) \stackrel{\iota_0}{\to} K_0(C(X_{\mathcal{P}(w)}^G) \rtimes G) \to \operatorname{Inv} C(Z_w, \mathbb{Z}) \to 0,$$

where up to the Morita equivalence $C(X^G_{\mathcal{P}(w)}) \rtimes G \cong C^*_r(\mathcal{G},\lambda) \rtimes_{\tilde{\alpha_w}} \mathbb{Z}$*, the element* $\iota_0[2^{-n}\chi_E]$ *is the image of* $[\chi_E \otimes \Theta_{f,f} \otimes \chi_{F_{n,0}}] \in K_0(C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z})$ *under the homomorphism induced in* K*-theory by the inclusion*

$$C(Z_w)\otimes\mathcal{K}\otimes C(\Omega)\rtimes\mathbb{Z} \cong C^*_r(\mathcal{G},\lambda) \hookrightarrow C^*_r(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_w}}\mathbb{Z},$$

where

- χ_E *is the characteristic function of a compact-open subset* E *of* Z_w*;*
- $\chi_{F_{n,0}}$ *is the characteristic function of* $F_{n,0} = 2^n\Omega$*;*
- $\Theta_{f,f}$ *is the rank one projector associated to a norm* 1 *function* f *of* $L^2([0,1])$*.*

Theorem 8.6. *We have an isomorphism*

$$\operatorname{Coinv} C(Z_w,\mathbb{Z}) \xrightarrow{\cong} K_1(C(X^G_{\mathcal{P}(w)})\rtimes G)$$

induced on the coinvariants by the composition

$$C(Z_w,\mathbb{Z}) \cong K_0(C(Z_w)) \xrightarrow{\otimes[u]} K_1(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z}) \cong K_1(C^*_r(\mathcal{G},\lambda) \to K_1(C^*_r(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_w}}\mathbb{Z}),$$

where

- $\otimes[u]$ *is the external product in* K*-theory by the class in* $K_1(C(\Omega)\rtimes\mathbb{Z})$ *of the unitary* u *of* $C(\Omega)\rtimes\mathbb{Z}$ *corresponding to the positive generator of* $\mathbb{Z}$*;*
- *the last map in the composition is the homomorphism induced in* K*-theory by the inclusion* $C^*_r(\mathcal{G},\lambda) \hookrightarrow C^*_r(\mathcal{G},\lambda)\rtimes_{\tilde{\alpha_w}}\mathbb{Z}$*.*

The short exact sequence of theorem 8.5, admits an explicit splitting which can be described in the following way: assume first that (Z_w,σ) is minimal. In particular, $\operatorname{Inv} C(Z_w,\mathbb{Z}) \cong \mathbb{Z}$ is generated by $1 \in C(Z_w,\mathbb{Z})$. Let us considerer the following diagram, whose left square is commutative

$$\begin{array}{ccccc} K_1(C^*(\mathbb{R})) & \longrightarrow & K_1(C^*(\mathcal{G},\lambda)) & & \\ \downarrow & & \downarrow & & \\ \mathbb{Z}\cong K_0(\mathbb{C}) & \longrightarrow & K_0(C(X^{\mathcal{N}}_{\mathcal{P}}\times Z_w)) & \xrightarrow{ev_*} & K_0(C(\Omega\times Z_w)) \end{array},$$

where

- the horizontal maps of the left square are induced by the inclusion $\mathbb{C} \hookrightarrow C(X^{\mathcal{N}}_{\mathcal{P}}\times Z_w)$.
- vertical maps are the Thom-Connes isomorphisms.
- The map $ev : C(X^{\mathcal{N}}_{\mathcal{P}}\times Z_w)) \longrightarrow C(\Omega\times Z_w)$ is induced by the continuous map $\Omega \to X^{\mathcal{N}}_{\mathcal{P}} \cong (\Omega\times\mathbb{R})/\mathcal{A}_o$; $x \mapsto [x,0]$;

Up to the Morita equivalence between $C^*(\mathcal{G},\lambda)$ and $C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z}$, the right down staircase is the boundary of the Pimsner-Voiculescu six-term exact sequence that computes $K_*(C(Z_w)\otimes C(\Omega)\rtimes\mathbb{Z})$. From this, we see that $K_1(C^*(\mathcal{G},\lambda)\cong\mathbb{Z}$ is generated by the image of the generator ζ of $K_1(C^*(\mathbb{R}))$ corresponding under the canonical identification $K_1(C^*(\mathbb{R})) \cong K_1(C_0(\mathbb{R})) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ to the class of any rank one projector in some $M_n(\mathbb{C})$. On the other hand, we have a diagram with

commutative squares

$$\begin{array}{ccccccc}
K_0(C^*(\mathbb{R})\rtimes\mathbb{R}_+^*) & \longrightarrow & K_0(C(X^G_{\mathcal{P}(w)})\rtimes G) & \xrightarrow[\cong]{} & K_0(C^*(\mathcal{G}_{\alpha_w},\lambda_{\alpha_w})\rtimes\mathbb{R}) & \xrightarrow[\cong]{} & K_0(C^*(\mathcal{G},\lambda)_{\alpha_w}\rtimes\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_1(C^*(\mathbb{R})) & \longrightarrow & K_1(C(X^G_{\mathcal{P}(w)})\rtimes\mathbb{R}) & \xrightarrow[\cong]{} & K_1(C^*(\mathcal{G}_{\alpha_w},\lambda_{\alpha_w})) & \xrightarrow[\cong]{} & K_1(C^*(\mathcal{G},\lambda)_{\alpha_w}) \\
 & & & & & & \downarrow ev_* \\
 & & & & & & K_0(C^*(\mathcal{G},\lambda))
\end{array}$$

where,

- the horizontal maps of the left square are induced by the inclusion $C^*(\mathbb{R}) \hookrightarrow C(X^G_{\mathcal{P}(w)})\rtimes\mathbb{R}$;
- the horizontal maps of the middle square are induced by the isomorphism of lemma 7.1
- the horizontal maps of the right square are induced by the isomorphism of proposition 5.7
- the first row of vertical maps are Thom-Connes isomorphisms.

It is then straightforward to check that the down staircase of the diagram is indeed induced by the inclusion $C^*(\mathbb{R}) \hookrightarrow C(X^{\mathcal{N}}_{\mathcal{P}}\times Z_w)\rtimes\mathbb{R} = C^*(\mathcal{G},\lambda)$. Notice that $K_0(C^*(\mathbb{R})\rtimes\mathbb{R}_+^*)\cong\mathbb{Z}$ (by Thom-Connes isomorphism). Moreover, under the inclusion $C_0(\mathbb{R}_+^*)\rtimes\mathbb{R}_+^* \hookrightarrow C_0(\mathbb{R})\rtimes\mathbb{R}_+^* \cong C^*(\mathbb{R})\rtimes\mathbb{R}_+^*$, any rang one projector e of $\mathcal{K}(L^2(\mathbb{R}_+^*)) \cong C_0(\mathbb{R}_+^*)\rtimes\mathbb{R}_+^*$ provides a generator for $K_0(C^*(\mathbb{R})\rtimes\mathbb{R}_+^*)$ whose image under the left vertical map is the generator ζ for $K_1(C^*(\mathbb{R}))\cong K_0(\mathbb{C})\cong\mathbb{Z}$. Using the description of the boundary map of Pimsner-Voiculescu six-term exact sequence, we see that e, viewed as an element of $C(X^G_{\mathcal{P}(w)})\rtimes G$ whose class in K-theory provides a lift for $1\in C(Z_w,\mathbb{Z})$ in the short exact sequence of theorem 8.5.

In general, $\mathrm{Inv}\,C(Z_w,\mathbb{Z})$ is generated by characteristic functions of $\mathbb{Z}$-invariant compact-open subsets of Z_w. According to proposition 4.3, any $\mathbb{Z}$-invariant compact-open subset E of Z_w provides a $\mathbb{R}$-invariant compact subset $\widetilde{E}$ of $X^G_{\mathcal{P}}$. Hence, with above notations, if $\chi_{\widetilde{E}}$ is the characteristic function for $\widetilde{E}$, then $\chi_{\widetilde{E}}e$ can be viewed as an element of $C(X^G_{\mathcal{P}(w)})\rtimes G$. Let $\upsilon : \mathrm{Inv}\,C(Z_w,\mathbb{Z}) \to K_0(C(X^G_{\mathcal{P}(w)})\rtimes G)$ be the group homomorphism uniquelly defined by $\upsilon(\chi_E) = \chi_{\widetilde{E}}e$ for E a $\mathbb{Z}$-invariant compact-open subset of Z_w. Then υ is a section for the short exact sequence of theorem 8.5.

9. Topological invariants for the continuous hull

It is known that for Euclidian tilings, topological invariants of the continuous hull are closely related to the K-theory of the C^*-algebra associated to the tiling. The K-theory of the latter turn out to be isomorphic to the K-theory of the hull which is using the Chern character rationally isomorphic to the integral Cĕch cohomology. Moreover, in dimension less or equal to 3, the Chern character can be defined valued in integral cohomology and we eventually obtain an isomorphism between the integral Cĕch cohomology of the hull and the K-theory of the C^*-algebra associated to the tiling. In consequence of this fact, a lot of interest has been generated in the computation of topological invariants of the hull.

For Penrose hyperbolic tilings, since the group of affine isometries of the hyperbolic half-plane is isomorphic to a semi-direct product $\mathbb{R} \rtimes \mathbb{R}$, we get using the Thom-Connes isomorphism that

$$K_*(C(X^G_{\mathcal{P}(w)}) \rtimes G) \cong K^*(X^G_{\mathcal{P}(w)}). \tag{9.1}$$

Moreover, since the cohomological dimension of $X^G_{\mathcal{P}(w)}$ is 2, the Chern character can also be defined with values in integral Cěch cohomology and hence we get as in the Euclidian case of low dimension an isomorphism

$$K_*(C(X^G_{\mathcal{P}(w)}) \rtimes G) \cong \check{H}(X^G_{\mathcal{P}(w)}, \mathbb{Z}).$$

These topological invariants can be indeed computed directly using technics very closed to those used in section 8. Indeed for a C^*-algebra A provided with an automorphism β, there is natural isomorphisms

$$K_0(A_\beta) \cong K_1(A \rtimes_\beta \mathbb{Z}) \quad \text{and} \quad K_1(A_\beta) \cong K_0(A \rtimes_\beta \mathbb{Z}), \tag{9.2}$$

called the mapping torus isomorphisms, where A_β is the mapping torus algebra constructed at the end of section 5.2. Recall from proposition 4.3 that $X^G_{\mathcal{P}(w)}$ can be viewed as a double suspension

$$(((\Omega \times \mathbb{R})/\mathcal{A}_o) \times Z_\omega \times \mathbb{R})/\mathcal{A}_f \tag{9.3}$$

with $f : (\Omega \times \mathbb{R})/\mathcal{A}_o \to (\Omega \times \mathbb{R})/\mathcal{A}_o$; $([x,t],\omega') \mapsto ([2x,2t],\sigma(\omega'))$. In regard of the mapping torus isomorphism, the double crossed product by $\mathbb{Z}$ corresponds in K-theory to the double suspension structure on $X^G_{\mathcal{P}(w)}$. For people interested in topological invariants, we explain how a straight computation can be carried out.

For a C^*-algebra A provided with an automorphism β, the short exact sequence

$$0 \to C_0((0,1), A) \to A_\beta \stackrel{ev}{\to} A \to 0, \tag{9.4}$$

where ev is the evaluation at 0 of elements of $A_\beta \subset C([0,1], A)$, gives rise to short exact sequences

$$0 \to \mathrm{Coinv}\, K_1(A) \to K_0(A_\beta) \to \mathrm{Inv}\, K_0(A) \to 0$$

and

$$0 \to \mathrm{Coinv}\, K_0(A) \to K_1(A_\beta) \to \mathrm{Inv}\, K_1(A) \to 0$$

where invariants and coinvariants are taken with respect to the action induced by β on $K_*(A)$ (see section 8). In particular, if X is a compact set and $f : X \to X$ is a homeomorphism, and with the notations of section 3.2 the mapping torus of $C(X)$ with respect to the automorphism induced by f is $C((X \times \mathbb{R})/\mathcal{A}_f)$. We deduce short exact sequences

$$0 \to \mathrm{Coinv}\, K^1(X) \to K^0((X \times \mathbb{R})/\mathcal{A}_f) \to \mathrm{Inv}\, K^0(X) \to 0$$

and

$$0 \to \mathrm{Coinv}\, K^0(X) \to K^1((X \times \mathbb{R})/\mathcal{A}_f) \to \mathrm{Inv}\, K^1(X) \to 0.$$

Similarly, we have short exact sequences in Cěch cohomology

$$0 \to \mathrm{Coinv}\, \check{H}^{n-1}(X, \mathbb{Z}) \to \check{H}^n((X \times \mathbb{R})/\mathcal{A}_f, \mathbb{Z}) \to \mathrm{Inv}\, \check{H}^n(X, \mathbb{Z}) \to 0,$$

derived from the inclusion

$$(0,1) \times X \hookrightarrow (X \times \mathbb{R})/\mathcal{A}_f. \tag{9.5}$$

Since the space $X^G_{\mathcal{P}(w)}$ has a structure of double suspension, we see following the same route as in section 8 that $X^G_{\mathcal{P}(w)}$ only has cohomology in degree 0, 1 and 2 and we get isomorphisms

$$K^0(X^G_{\mathcal{P}(w)}) \cong \mathrm{Inv}\, C(Z_w,\mathbb{Z}) \oplus \mathrm{Coinv}\, C(Z_w,\mathbb{Z}[1/2]) \tag{9.6}$$

$$K^1(X^G_{\mathcal{P}(w)}) \cong \mathrm{Coinv}\, C(Z_w,\mathbb{Z}) \tag{9.7}$$

$$\check{H}^0(X^G_{\mathcal{P}(w)},\mathbb{Z}) \cong \mathrm{Inv}\, C(Z_w,\mathbb{Z}) \tag{9.8}$$

$$\check{H}^1(X^G_{\mathcal{P}(w)},\mathbb{Z}) \cong \mathrm{Coinv}\, C(Z_w,\mathbb{Z}) \tag{9.9}$$

$$\check{H}^2(X^G_{\mathcal{P}(w)},\mathbb{Z}) \cong \mathrm{Coinv}\, C(Z_w,\mathbb{Z}[1/2]). \tag{9.10}$$

Recall that invariants and coinvariants of $C(Z_w,\mathbb{Z})$ are taken with respect to the automorphism

$$C(Z_w,\mathbb{Z}) \to C(Z_w,\mathbb{Z});\ f \mapsto f\circ\sigma^{-1},$$

and that coinvariants of $C(Z_w,\mathbb{Z}[1/2])$ are taken with respect to the automorphism

$$C(Z_w,\mathbb{Z}[1/2]) \to C(Z_w,\mathbb{Z}[1/2]);\ f \mapsto 2f\circ\sigma^{-1}.$$

Let us describe explicitly these isomorphisms. The identification of equation (9.3) yields to a continuous map

$$X^G_{\mathcal{P}(w)} \to (Z_w\times\mathbb{R})/\mathcal{A}_\sigma, \tag{9.11}$$

induced by the equivariant projection $((\Omega\times\mathbb{R})/\mathcal{A}_o)\times Z_\omega\times\mathbb{R}\to Z_w\times\mathbb{R}$. Together with the inclusion $Z_w\times(0,1)\hookrightarrow(Z_w\times\mathbb{R})/\mathcal{A}_\sigma$, this gives rise to a homomorphism $K^1(Z_w\times(0,1))\to K^1(X^G_{\mathcal{P}(w)})$ inducing under Bott peridodicity the isomorphism of equation (9.7) (recall from lemma 8.1 that $K^0(Z_w)=K_0(C(Z_w))\cong C(Z_\omega,\mathbb{Z})$). The identification of equation (9.9) is obtained in the same way by using the isomorphism

$$C(Z_\omega,\mathbb{Z})\cong\check{H}^0(Z_\omega,\mathbb{Z})\xrightarrow{\cong}\check{H}^1(Z_\omega\times(0,1),\mathbb{Z})$$

provided by the cup product by the fundamental class of $\check{H}^1((0,1),\mathbb{Z})$. Recall that $\mathrm{Inv}\, C(Z_w,\mathbb{Z})$ is generated by characteristic functions of invariant compact-open subsets of Z_ω. If E is such a subset, then $(E\times\mathbb{R})/\mathcal{A}_{\sigma_{/E}}$ is a compact-open subset of $(Z_w\times\mathbb{R})/\mathcal{A}_\sigma$ and is pulled-back under the map of equation (9.11) to a compact-open subset $\widetilde{E}$ of $X^G_{\mathcal{P}(w)}$. The isomorphism of equation (9.8) identifies $\chi_E\in\mathrm{Inv}\, C(\mathbb{Z}_\omega,\mathbb{Z})$ with the class of $\widetilde{E}$ in $\check{H}^0(X^G_{\mathcal{P}(w)},\mathbb{Z})$ and the isomorphism of equation (9.6) identifies χ_E with the class of $\chi_{\widetilde{E}}$ in $K^0(Z_w)=K_0(C(Z_w))$. Using twice the inclusion of equation (9.5) for the double suspension structure of $X^G_{\mathcal{P}(w)}$, we obtain an inclusion $\Omega\times Z_\omega\times(0,1)^2\hookrightarrow X^G_{\mathcal{P}(w)}$ and hence by Bott periodicity a map $C(\Omega\times Z_\omega,\mathbb{Z})\to K^0(X^G_{\mathcal{P}(w)})$. Then, if E is a compact-open subset of Z_ω and n is an integer, the image of $\chi_{E\times 2^n\Omega}$ under this map is up to identification of equation (9.6) the class of $\chi_E/2^n$ in $\mathrm{Coinv}\, C(Z_w,\mathbb{Z}[1/2])$. The description of the identification of equation (9.10) is obtained in the same way by using the isomorphism

$$C(\Omega\times Z_\omega,\mathbb{Z})\cong\check{H}^0(\Omega\times Z_\omega,\mathbb{Z})\xrightarrow{\cong}\check{H}^2(\Omega\times Z_\omega\times(0,1)^2,\mathbb{Z})$$

provided by the cup product by the fundamental class of $\check{H}^2((0,1)^2,\mathbb{Z})$. Moreover, since the Chern character is natural and intertwins Bott periodicity and the cup product by the fundamental class of $\check{H}^1_c((0,1),\mathbb{Z})$, we deduce that up to the identifications of equations (9.6) to (9.10), it is given by the identity maps of $\mathrm{Inv}\, C(Z_w,\mathbb{Z})\oplus\mathrm{Coinv}\, C(Z_w,\mathbb{Z}[1/2])$ and of $\mathrm{Coinv}\, C(Z_w,\mathbb{Z})$. It is easy to guess how

the generators of $K^*(X^G_{\mathcal{P}(w)})$ described in equations (9.6) and (9.7) should be identified with those of $K_*(C(X^G_{\mathcal{P}(w)})\rtimes G)$ described in section 8 under Thom-Connes isomorphism of equation (9.2). Recall first that for a unital C^*-algebra A provided with an automorphism β,

- the mapping torus A_β is provided with an action $\widehat{\beta}$ of $\mathbb{R}$ by automorphisms (see section 5.2) and moreover $A\rtimes_\beta\mathbb{Z}$ and $A_\beta\rtimes_{\widehat{\beta}}\mathbb{R}$ are Morita equivalent;
- the mapping torus isomorphisms are the composition of the Thom-Connes isomorphisms $K_0(A_\beta)\xrightarrow{\cong}K_1(A\rtimes_\beta\mathbb{Z})$ and $K_1(A_\beta)\xrightarrow{\cong}K_0(A\rtimes_\beta\mathbb{Z})$ with the isomorphism $K_*(A_\beta\rtimes_{\widehat{\beta}}\mathbb{R})\cong K_*(A\rtimes_\beta\mathbb{Z})$ induced with the above Morita-equivalence;

It is then straightforward to check that viewing $C(X^G_{\mathcal{P}(w)})\rtimes G$ as a double crossed product by $\mathbb{Z}$ as we did in section 7, the Thom-Connes isomorphism

$$K_*(C(X^G_{\mathcal{P}(w)}))\stackrel{\cong}{\to}K_*(C(X^G_{\mathcal{P}(w)})\rtimes G)$$

is obtained by using twice the mapping torus isomorphim (up to stabilisation for the second one). In view of our purpose of idenfying the generators of $K_*(C(X^G_{\mathcal{P}(w)}))$ with those of $K_*(C(X^G_{\mathcal{P}(w)})\rtimes G)$, we will need the following alternative description of the mapping torus isomorphism using the bivariant Kasparov K-theory groups $KK^{\mathbb{Z}}_*(\bullet,\bullet)$ [11]. Let A be a C^*-algebra and let β be an automorphism of A. Since the action of $\mathbb{Z}$ on $\mathbb{R}$ by translations is free and proper, we have a Morita equivalence between A_β and $C_0(\mathbb{R},A)\rtimes\mathbb{Z}$, where $C_0(\mathbb{R},A)\cong C_0(\mathbb{R})\otimes A$ is equipped with the diagonal action of $\mathbb{Z}$. Recall that the $\mathbb{Z}$-equivariant unbounded operator $\imath\frac{d}{dt}$ of $L^2(\mathbb{R})$ gives rise to a unbounded K-cycle and hence to an element y in $KK^{\mathbb{Z}}_1(C_0(\mathbb{R}),\mathbb{C})$. Then the mapping torus isomorphism of equation (9.2) is the composition

$$K_*(A_\beta)\xrightarrow{\cong}K_*(C_0(\mathbb{R},A)\rtimes\mathbb{Z})\stackrel{\otimes_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}}J_{\mathbb{Z}}(\tau_A(y))}{\longrightarrow}K_{*+1}(A\rtimes_\beta\mathbb{Z})$$

where,

- the first map comes from the Morita equivalence;
- $J_{\mathbb{Z}}: KK^{\mathbb{Z}}_*(\bullet,\bullet)\to KK_*(\bullet\rtimes\mathbb{Z},\bullet\rtimes\mathbb{Z})$ is the Kasparov transformation in bivariant KK-theory;
- for any C^*-algebra B equipped with an action of Γ by automorphism τ_B : $KK^\Gamma_*(\bullet,\bullet)\to KK^\Gamma_*(\bullet\otimes B,\bullet\otimes B)$ is the tensorisation operation;
- $\otimes_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}}J_{\mathbb{Z}}(\tau_A(y))$ stands for the right Kasparov product by $J_{\mathbb{Z}}(\tau_A(y))$.

Then the identification between the generators of $K^*(X^G_{\mathcal{P}(w)})$ and of $K_*(C(X^G_{\mathcal{P}(w)})\rtimes G)$ can be achieved using the next two lemmas.

Lemma 9.1. *Let A be a unital C^*-algebra together with an automorphism β. Let e be invariant projector in A and let x_e be the class in $K_0(A_\beta)$ of the projector $[0,1]\to A$; $t\mapsto e$. Then the image of x_e under the mapping torus isomorphism is equal to the class of the unitary $1-e+e\cdot u$ of $A\rtimes_\beta\mathbb{Z}$ in $K_1(A\rtimes_\beta\mathbb{Z})$ (here u is the unitary of $A\rtimes_\beta\mathbb{Z}$ corresponding to the positive generator of $\mathbb{Z}$);*

Proof. The invariant projector e gives rise to an equivariant map $\mathbb{C}\to A$; $z\to ze$ and hence to a homomorphism $C(\mathbb{T})\to A_\beta$. By naturality of the mapping torus, this amounts to prove the result for $A=\mathbb{C}$ which is done in [23, Example 6.1.6]. □

Lemma 9.2. *Let A be a unital C^*-algebra together with an automorphism β and let x be an element in $K_*(A)$. The two following elements then coincide:*

- *the image of x under the composition*

$$K_*(A) \stackrel{\cong}{\to} K_{*+1}(C((0,1),A)) \to K_{*+1}(A_\beta) \to K_{*+1}(A \rtimes_\beta \mathbb{Z}),$$

  where
  - *the first map is the Bott periodicity isomorphism;*
  - *the second map is induced by the inclusion $C((0,1),A) \hookrightarrow A_\beta$;*
  - *the third map is the mapping torus isomorphism.*
- *the image of x under the map $K_*(A) \to K_*(A \rtimes_\beta \mathbb{Z})$ induced by the inclusion $A \hookrightarrow A \rtimes_\beta \mathbb{Z}$.*

Proof. Let us first describe the imprimity bimodule implementing the Morita equivalence between A_β and $C_0(\mathbb{R}, A) \rtimes \mathbb{Z}$. Indeed, in a more general setting, if

- X is a locally compact space equipped with a proper action of $\mathbb{Z}$ by homeomorphisms,
- B is a C^*-algebra provided with an action of $\mathbb{Z}$ by automorphisms;
- $B_X^{\mathbb{Z}}$ stands for the algebra of equivariant continuous maps $f : X \to B$ such that $\mathbb{Z}.x \mapsto \|f(x)\|$ belongs to $C_0(X/\mathbb{Z})$.

then, if we equip $C_0(X,B) \cong C_0(X) \otimes B$ with the diagonal action of $\mathbb{Z}$, there is an imprimitivity $B_X^{\mathbb{Z}} - C_0(X,B) \rtimes \mathbb{Z}$-bimodule defined in the following way: let us consider on $C_c(X,B)$ the $C_0(X,B) \rtimes \mathbb{Z}$-valued inner product

$$\langle \xi, \xi' \rangle(n) = n(\xi^*)\xi',$$

for ξ and ξ' in $C_c(X,B)$ and n in $\mathbb{Z}$. This inner product is namely positive and gives rise to a right $C_0(X,B) \rtimes \mathbb{Z}$-Hilbert module $\mathcal{E}(B,X)$, the action of $C_0(X,B) \rtimes \mathbb{Z}$ on the right being given for ξ in $C_c(X,B)$ and h in $C_c(\mathbb{Z} \times X, B) \subset C_c(\mathbb{Z}, C_0(X,B))$ by

$$\xi \cdot h(t) = \sum_{n \in \mathbb{Z}} n(\mathcal{E}(n+t)) n(h(n, n+t)).$$

The action by pointwise multiplication of $B_X^{\mathbb{Z}} \subset C_b(X,B)$ on $C_0(X,B)$ extends to a left $B_X^{\mathbb{Z}}$-module structure on $\mathcal{E}(B,X)$. Let us denote by $[\mathcal{E}(B,X)]$ the class of the $B_X^{\mathbb{Z}} - C_0(X,B) \rtimes \mathbb{Z}$-bimodule $\mathcal{E}(B,X)$ in $KK_*(B_X^{\mathbb{Z}}, C_0(X,B) \rtimes \mathbb{Z})$. It is straightforward to check that

- $[\mathcal{E}(B,X)]$ is natural in both variable, in particular, if Y is an open invariant subset of X and let us denote by $\iota_{Y,X,B} : C_0(Y,B) \to C_0(X,B)$ and $\iota^{\mathbb{Z}}_{Y,X,B} : B_Y^{\mathbb{Z}} \to B_X^{\mathbb{Z}}$ the homomorphisms induced by the inclusion $Y \hookrightarrow X$ and respectively by $[\iota_{Y,X,B}]$ and $[\iota^{\mathbb{Z}}_{Y,X,B}]$ the corresponding classes in $KK^{\mathbb{Z}}_*(C_0(Y,B), C_0(X,B))$ and $KK_*(B_Y^{\mathbb{Z}}, B_X^{\mathbb{Z}})$, then

$$[\iota^{\mathbb{Z}}_{Y,X,B}] \otimes_{B_X^{\mathbb{Z}}} [\mathcal{E}(B,X)] = [\mathcal{E}(B,Y)] \otimes_{C_0(Y,B)} J_{\mathbb{Z}}([\iota_{Y,X,B}]).$$

- up to the identification $B_{\mathbb{Z}}^{\mathbb{Z}} \cong B$, the class $[\mathcal{E}(B,\mathbb{Z})]$ is induced by the composition

$$B \longrightarrow C_0(\mathbb{Z}, B) \hookrightarrow C_0(\mathbb{Z}, B) \rtimes \mathbb{Z}$$

  where the first map is $b \mapsto \delta_0 \otimes b$.
- if V is any locally compact space, and if we consider $\mathbb{Z}$ acting trivially on it, then up to the identifications $B^{\mathbb{Z}}_{X \times V} \cong B^{\mathbb{Z}}_X \otimes C_0(V)$ and $C_0(X \times V, B) \rtimes \mathbb{Z} \cong C_0(X,B) \rtimes \mathbb{Z} \otimes C_0(V)$, we have $[\mathcal{E}(B, V \times X)] = \tau_{C_0(V)}([\mathcal{E}(B,X)])$.

Noticing that for a C^*-algebra A provided with an automorphism β, we have a natural identification $A_\beta \cong A^{\mathbb{Z}}_{\mathbb{R}}$, the mapping torus isomorphism of equation (9.2) is obtained by right Kasporov product with $[\mathcal{E}(A,\mathbb{R})] \otimes_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}} J_{\mathbb{Z}}(\tau_A([y]))$. From this, we see that the composition in the statement of the lemma is given by right Kasparov product with

$$z = \tau_A([\partial]) \otimes_{C_0((0,1),A)} [\iota^{\mathbb{Z}}_{(0,1)\times\mathbb{Z},\mathbb{R},A}] \otimes_{A_\beta} [\mathcal{E}(A,\mathbb{R})] \otimes_{C_0(\mathbb{R},A)\rtimes\mathbb{Z}} J_{\mathbb{Z}}(\tau_A(y)) \tag{9.12}$$

where

- $[\partial]$ in $KK_1(\mathbb{C}, C_0(0,1))$ is the boundary of the evaluation at 0 extension
$$0 \to C_0(0,1) \to C_0[0,1) \to \mathbb{C} \to 0;$$
- we have used the identification $(0,1)\times\mathbb{Z} \cong \mathbb{R}\setminus\mathbb{Z}$ to see $(0,1)\times\mathbb{Z}$ as an invariant open subset of $\mathbb{R}$, in particular we have $A^{\mathbb{Z}}_{(0,1)\times\mathbb{Z}} \cong C_0((0,1),A)$.

According to the naturality properties of $[\mathcal{E}(\bullet,A)]$ listed above, we get that

$$\begin{aligned} [\iota^{\mathbb{Z}}_{(0,1)\times\mathbb{Z},\mathbb{R},A}] \otimes_{A_\beta} [\mathcal{E}(\mathbb{R},A)] &= [\mathcal{E}((0,1)\times\mathbb{Z},A)] \otimes_{C_0(\mathbb{Z},A)\rtimes\mathbb{Z}\otimes C_0(0,1)} J_{\mathbb{Z}}([\iota_{(0,1)\times\mathbb{Z},\mathbb{R},A}]) \\ &= \tau_{C_0(0,1)}([\mathcal{E}(\mathbb{Z},A)]) \otimes_{C_0(\mathbb{Z},A)\rtimes\mathbb{Z}\otimes C_0(0,1)} J_{\mathbb{Z}}([\iota_{(0,1)\times\mathbb{Z},\mathbb{R},A}]) \end{aligned} \tag{9.13}$$

Using commutativity of exterior Kasparov product, we get from equation (9.13) that

$$z = [\mathcal{E}(\mathbb{Z},A)] \otimes_{C_0(\mathbb{Z},A)\rtimes\mathbb{Z}} J_{\mathbb{Z}}(\tau_{C_0(\mathbb{Z},A)}([\partial]) \otimes_{C_0((0,1)\times\mathbb{Z},A)} [\iota_{(0,1)\times\mathbb{Z},\mathbb{R},A}] \otimes \tau_A(y)). \tag{9.14}$$

Let y' be the element of $KK_1(C_0(0,1),\mathbb{C})$ corresponding in the unbounded picture to the unbounded operator $\imath\frac{d}{dt}$ on $L^2(0,1)$ and let $[\mathcal{F}]$ be the element of $KK^{\mathbb{Z}}(C_0(\mathbb{Z}),\mathbb{C})$ corresponding to the equivariant representation by compact operator of $C_0(\mathbb{Z})$ onto $\ell^2(\mathbb{Z})$ (equipped with the left regular representation) given by pointwise multiplication. Then it is straightforward to check that

$$[\iota_{(0,1)\times\mathbb{Z},\mathbb{R},A}] \otimes_{C_0(\mathbb{R},A)} \tau_A(y) = \tau_A(\tau_{C_0(\mathbb{Z})}(y') \otimes_{C_0(\mathbb{Z})} [\mathcal{F}]).$$

But it is a standard fact that $[\partial] \otimes_{C_0(0,1)} y' = 1$ in the ring $KK_0(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}$ and hence we eventually get that

$$z = [\mathcal{E}(A,\mathbb{Z})] \otimes_{C_0(\mathbb{Z},A)\rtimes\mathbb{Z}} J_{\mathbb{Z}}(\tau_A([\mathcal{F}])).$$

A direct inspection of the right hand side of this equality shows that z is indeed the class of $KK_*(A, A\rtimes\mathbb{Z})$ induced by the inclusion $A \hookrightarrow A\rtimes\mathbb{Z}$. □

Recall that in section 8, we have established isomorphisms

$$K_0(C(X^G_{\mathcal{P}(w)}) \rtimes G) \cong \mathrm{Coinv}\, C(Z_w, \mathbb{Z}[1/2]) \oplus \mathrm{Inv}\, C(Z_w,\mathbb{Z}) \tag{9.15}$$

and

$$K_1(C(X^G_{\mathcal{P}(w)}) \rtimes G) \cong \mathrm{Coinv}\, C(Z_w,\mathbb{Z}). \tag{9.16}$$

Under this identification, and using lemma 9.1 and twice lemma 8.2, we are now in position to describe the image of the generators of $K^*(X^G_{\mathcal{P}(w)})$ under the double Thom-Connes isomorphism.

Corollary 9.3. *Under the identification of equations (9.6), (9.7), (9.15) and (9.16), the double Thom-Connes isomorphism*

$$K^*(X^G_{\mathcal{P}(w)}) \xrightarrow{\cong} K_*(C(X^G_{\mathcal{P}(w)}) \rtimes G)$$

corresponds to the identity maps of $\mathrm{Coinv}\, C(Z_w,\mathbb{Z}[1/2]) \oplus \mathrm{Inv}\, C(Z_w,\mathbb{Z})$ *and* $\mathrm{Coinv}\, C(Z_w,\mathbb{Z})$.

Proof. The statement concerning the factor $\mathrm{Inv}\, C(Z_w, \mathbb{Z})$ is indeed a consequence of the discussion at the end of section 8. Before proving the statements concerning the factors $\mathrm{Coinv}\, C(Z_w, \mathbb{Z})$ and $\mathrm{Coinv}\, C(Z_w, \mathbb{Z}[1/2])$, we sum up for convenience of the reader the main features, described in sections 4 and 7 of the dynamic of the continuous hull for the coloured and the uncoloured Penrose hyperbolic tilings.

- the closure $X_{\mathcal{P}}^{\mathcal{N}}$ of $\mathcal{N}_{\mathcal{P}} \cdot \mathcal{P}$ in $X_{\mathcal{P}}^{G}$ for the tiling topolology of the Penrose hyperbolic tiling $\mathcal{P}$ is homeomorphic to the suspension of the homeomorphism $o : \Omega \to \Omega$; $\omega \mapsto \omega + 1$;
- the continuous hull $X_{\mathcal{P}(w)}^{G}$ of the coloured Penrose hyperbolic tiling $\mathcal{P}(w)$ is homeomorphic to the suspension of the homeomorphism $X_{\mathcal{P}}^{\mathcal{N}} \times Z_\omega \to X_{\mathcal{P}}^{\mathcal{N}} \times Z_\omega : (\mathcal{T}, w') \mapsto (R \cdot \mathcal{T}, \sigma(w'))$;
- If we provide $X_{\mathcal{P}}^{\mathcal{N}} \times Z_\omega$ with the diagonal action of $\mathbb{R}$, by translations on $X_{\mathcal{P}}^{\mathcal{N}}$ and trivial on Z_ω, and equip the groupoid $\mathcal{G} = (X_{\mathcal{P}}^{\mathcal{N}} \times Z_\omega) \rtimes \mathbb{R}$ with the Haar system arising from the Haar measure of $\mathbb{R}$, then $C(X_{\mathcal{P}(w)}^{G}) \rtimes \mathbb{R}$ is the mapping torus of $C_r^*(\mathcal{G}, \lambda)$ with respect to the automorphim $\tilde{\alpha_\omega}$ arising from the automorphism of groupoid $\mathcal{G} \to \mathcal{G}$; $(\mathcal{T}, w', t) \mapsto (R \cdot \mathcal{T}, \sigma(\omega'), 2t)$ (see section 7);
- $C_r^*(\mathcal{G}, \lambda)$ is Morita-equivalent to the crossed product $C(\Omega \times Z_\omega) \rtimes \mathbb{Z}$ for the action of $\mathbb{Z}$ on $C(\Omega \times Z_\omega)$ arising from $o \times Id_{Z_\omega}$ (see section 6).

Let us consider the following diagram:
(9.17)

$$\begin{array}{ccccc}
 & & K_i(C_r^*(\mathcal{G}, \lambda)) & \longrightarrow & K_i(C(X_{\mathcal{P}(w)}^{G}) \rtimes G) \\
 & & = \uparrow & & \uparrow TC \\
K_i(C(\Omega \times Z_w)) & \longrightarrow & K_i(C_r^*(\mathcal{G}, \lambda)) & \longrightarrow & K_{i+1}(C(X_{\mathcal{P}(w)}^{G}) \rtimes \mathbb{R}), \\
= \uparrow & & \uparrow TC & & \uparrow TC \\
K_i(C(\Omega \times Z_w)) & \longrightarrow & K_{i+1}(C(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w)) & \longrightarrow & K_i(C(X_{\mathcal{P}(w)}^{G}))
\end{array}$$

where

- the bottom and the right middle horizontal arrows are the maps defined for any C^*-algebra A provided by an automorphism β as the composition
$$K_i(A) \to K_{i+1}(C_0((0,1), A) \to K_{i+1}(A_\beta)$$
of the Bott peridocity isomorphism with the homomorphism induced in K-theory by the inclusion $C_0((0,1), A) \hookrightarrow A_\beta$;
- the left middle horizontal arrow is up to the Morita equivalence between $C_r^*(\mathcal{G}, \lambda)$ and $C(\Omega \times Z_\omega) \rtimes \mathbb{Z}$ induced by the inclusion $C(\Omega \times Z_\omega) \hookrightarrow C(\Omega \times Z_\omega) \rtimes \mathbb{Z}$;
- the top horizontal arrow is up to the Morita equivalence between $C_r^*(\mathcal{G}, \lambda) \rtimes_{\tilde{\alpha_\omega}} \mathbb{Z}$ and $C(X_{\mathcal{P}(w)}^{G}) \rtimes G$ is induced by the inclusion $C_r^*(\mathcal{G}, \lambda) \hookrightarrow C_r^*(\mathcal{G}, \lambda) \rtimes_{\tilde{\alpha_\omega}} \mathbb{Z}$.
- the vertical maps TC stand for the Thom-Connes isomorphisms.

Then the inclusion $\mathrm{Coinv}\, C(Z_w, \mathbb{Z}[1/2]) \hookrightarrow K^0(X_{\mathcal{P}(w)}^{G})$ of equation (9.6) is induced by the composition of the bottom arrows and the inclusion $\mathrm{Coinv}\, C(Z_w, \mathbb{Z}[1/2]) \hookrightarrow K_0(C(X_{\mathcal{P}(w)}^{G}) \rtimes G)$ of equation (9.15) is induced by the upper staircase. According to lemma 9.1, the left bottom and the right top squares are commutative. Hence, the proof of the statements regarding to the $\mathrm{Coinv}\, C(Z_w, \mathbb{Z}[1/2])$ summand amounts

to show that the bottom right square is commutative. To see this, let us equip $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1)$ with the action of $\mathbb{R}$ by homeomorphisms

$$X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1) \times \mathbb{R} \longrightarrow X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1); (\mathcal{T}, \omega', s, t) \mapsto (\mathcal{T} + 2^{-s}t, \omega', s).$$

If we restrict this action to $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1)$, then the inclusion $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1) \hookrightarrow X_{\mathcal{P}(w)}^{G}$ is $\mathbb{R}$-equivariant and the Bott periodicity isomorphism is the boundary of the equivariant short exact sequence

$$0 \to C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1)) \to C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1)) \to C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w) \to 0 \tag{9.18}$$

provided by evaluation at 0. This equivariant short exact sequence gives rise to a short exact sequence for crossed products

$$0 \to C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1)) \rtimes \mathbb{R} \to C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1)) \rtimes \mathbb{R} \to C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w) \rtimes \mathbb{R} \to 0. \tag{9.19}$$

and since the Thom-Connes isomorphism is natural, it intertwins the corresponding boundary maps and hence we get a commutative diagram

$$\begin{array}{ccccc} K_{i+1}(C(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w) \rtimes \mathbb{R}) & \longrightarrow & K_i(C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1) \rtimes \mathbb{R}) & \longrightarrow & K_i(C(X_{\mathcal{P}(w)}^{G}) \rtimes \mathbb{R}) \\ {\scriptstyle TC}\uparrow & & \uparrow{\scriptstyle TC} & & \uparrow{\scriptstyle TC} \\ K_i(C(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w)) & \longrightarrow & K_{i+1}(C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1)) & \longrightarrow & K_{i+1}(C(X_{\mathcal{P}(w)}^{G})) \end{array}, \tag{9.20}$$

where the left horizontal arrows are induced by the boundary maps corresponding to the exact sequences of equation (9.18) and (9.19) and the right horizontal arrows are induced by the equivariant inclusion $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1) \hookrightarrow X_{\mathcal{P}(w)}^{G}$. Let us consider the family groupoids $(0,1) \times \mathcal{G}$ and $[0,1) \times \mathcal{G}$. Notice that if $X_{\mathcal{P}(w)}^{G} \rtimes \mathbb{R}$ is viewed as the suspension of the groupoid $\mathcal{G}$, then $(0,1) \times \mathcal{G}$ is the restriction of $X_{\mathcal{P}(w)}^{G} \rtimes \mathbb{R}$ to a fundamental domain. The reduced C^*-algebras of these two groupoids are respectively $C_0((0,1), C_r^*(\mathcal{G}, \lambda)))$ and $C_0([0,1), C_r^*(\mathcal{G}, \lambda))$ and the automorphism of groupoids

$$[0,1) \times \mathcal{G} \to (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1)) \rtimes \mathbb{R}; (\mathcal{T}, \omega', s, t) \mapsto (\mathcal{T}, \omega', s, 2^s t)$$

gives rise to a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0((0,1), C_r^*(\mathcal{G}, \lambda)) & \longrightarrow & C_0([0,1), C_r^*(\mathcal{G}, \lambda)) & \longrightarrow & C_r^*(\mathcal{G}, \lambda) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times (0,1)) \rtimes \mathbb{R} & \longrightarrow & C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times [0,1)) \rtimes \mathbb{R} & \longrightarrow & C_0(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w) \rtimes \mathbb{R} & \longrightarrow & 0. \end{array}$$

Using naturality of the boundary map, we see that the composition of the top horizontal arrows in diagram (9.20) is the composition

$$K_i(C_r^*(\mathcal{G}, \lambda)) \to K_{i+1}(C_0((0,1), C_r^*(\mathcal{G}, \lambda)) \to K_{i+1}(C(X_{\mathcal{P}(w)}^{G}) \rtimes \mathbb{R}))$$

of the Bott peridocity isomorphism with the homomorphism induced in K-theory by the inclusion $C_0((0,1), C_r^*(\mathcal{G}, \lambda)) \hookrightarrow C(X_{\mathcal{P}(w)}^{G}) \rtimes \mathbb{R}$. This concludes the proof for the statement concerning the summand $\operatorname{Coinv} C(Z_\omega, \mathbb{Z}[1/2])$. The statement concerning the summand $\operatorname{Coinv} C(Z_\omega, \mathbb{Z})$ is a consequence of the commutativity of the top square of diagram 9.17 and of lemma 9.1 applied to the middle bottom vertical arrow. □

10. The cyclic cocycle associated to a harmonic probabilty

Recall that according to the discussion ending section 4.2, a probability is harmonic if and only if it is G-invariant. In this section, we associate to a harmonic probability a 3-cyclic cocycle on the smooth crossed product algebra of $X^G_{\mathcal{P}(\omega)} \rtimes G$. This cyclic cocycle is indeed builded from a 1-cyclic cocycle on the algebra of smooth (along the leaves) functions on $X^G_{\mathcal{P}(\omega)}$ by using the analogue in cyclic cohomology of the Thom-Connes isomorphism (see [6]). We give a description of this cocycle and we discuss an odd version of the gap-labelling.

10.1. **Review on smooth crossed products.** We collect here results from [6] concerning smooth crossed products by an action of $\mathbb{R}$ that we will need later on.

Let $\mathcal{A}$ be a Frechet algebra with respect to an increasing family of semi-norms $(\| \bullet \|_k)_{k\in\mathbb{N}}$.

Definition 10.1. *A smooth action on $\mathcal{A}$ is a homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}\ \mathcal{A}$ such that*

1. *For every t in $\mathbb{R}$ and a in $\mathcal{A}$, the function $t \mapsto \alpha_t(a)$ is smooth.*
2. *For every integers k and m, there exist integers j and n and a real C such that $\left\| \frac{d^k}{dt^k}\alpha_t(a) \right\|_m \leq C(1+t^2)^{j/2}\|a\|_n$ for every a in $\mathcal{A}$.*

If α is a smooth action on $\mathcal{A}$, then the smooth crossed product $\mathcal{A}\rtimes_\alpha\mathbb{R}$ is defined as the set of smooth functions $f : \mathbb{R} \to \mathcal{A}$ such that

$$\|f\|_{k,m,n} \stackrel{\mathrm{def}}{=\!=} \sup_{t\in\mathbb{R}} (1+t^2)^{k/2} \left\| \frac{d^m}{dt^m} f(t) \right\|_n < +\infty$$

for all integers k, m and n. The smooth crossed product $\mathcal{A}\rtimes_\alpha\mathbb{R}$ provided with the family of semi-norm $\| \bullet \|_{k,m,n}$ for k, m and n integers together with the convolution product

$$f * g(t) = \int f(s)\alpha_s(g(t-s))dt$$

is then a Frechet algebra. Notice that a smooth action α on a Frechet algebra $\mathcal{A}$ gives rise to a bounded derivation Z_α of $\mathcal{A}\rtimes_\alpha\mathbb{R}$ defined by $Z_\alpha(f)(t) = tf(t)$ for all f in $\mathcal{A}\rtimes_\alpha\mathbb{R}$ and t in $\mathbb{R}$.

Let $\mathcal{A}^G_{\mathcal{P}(\omega)}$ be the algebra of continuous and smooth along the leaves functions on $X^G_{\mathcal{P}(\omega)}$, i.e functions whose restrictions to leaves admit at all order differential which are continuous as functions on $X^G_{\mathcal{P}(\omega)}$. Let β^0 and β^1 be the two actions of $\mathbb{R}$ on $\mathcal{A}^G_{\mathcal{P}(\omega)}$ respectively induced by

$$\mathbb{R} \times X^G_{\mathcal{P}(\omega)} \to X^G_{\mathcal{P}(\omega)};\ (t, \mathcal{T}) \mapsto \mathcal{T} + t$$

and

$$\mathbb{R} \times X^G_{\mathcal{P}(\omega)} \to X^G_{\mathcal{P}(\omega)};\ (t, \mathcal{T}) \mapsto R_{2^t} \cdot \mathcal{T}.$$

Let X and Y be respectively the vector fields associated to β^0 and β^1. Then $\mathcal{A}^G_{\mathcal{P}(\omega)}$ is a Frechet algebra with respect to the family of semi-norms

$$f \mapsto \sup_{X^G_{\mathcal{P}(\omega)}} |X^k Y^l(f)|,$$

where k and l run through integers. It is clear that β^0 is a smooth action on $\mathcal{A}^G_{\mathcal{P}(\omega)}$. Moreover,

$$\mathbb{R} \times \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \to \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}; (t, f) \mapsto [s \mapsto \beta^1(f(2^{-t}s))]$$

is an action of $\mathbb{R}$ on $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$ by automorphisms. This action is not smooth in the previous sense. Nevertheless, the action β^1 satisfies conditions (1),(2) and (3) of [6, Section 7.2] with respect to the family of functions $\rho_n : \mathbb{R} \to \mathbb{R}; t \mapsto 2^{2n|t|}$, where n runs through integers. In this situation, we can define the smooth crossed product $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^{\rho}_{\beta^1} \mathbb{R}$ of $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$ by β^1 to be the set of smooth functions $f : \mathbb{R} \to \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$ such that

$$\|f\|_{k,l,m} \stackrel{\text{def}}{=\!=} \sup_{t \in \mathbb{R}} \rho_k(t) \left\| \frac{d^l}{dt^l} f(t) \right\|_m < +\infty$$

for all integers k, l and m (we have reindexed for convenience the family of semi-norms on $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$ using integers). Then $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^{\rho}_{\beta^1} \mathbb{R}$ provided with the family of semi-norms $\| \bullet \|_{k,l,m}$ for k, l and m integers together with the convolution product is a Frechet algebra. Moreover, this algebra can be viewed as a dense subalgebra of $C(X^G_{\mathcal{P}(\omega)}) \rtimes G$. As for smooth actions, the action β^1 gives rise to a derivation Z_{β^1} of $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^{\rho}_{\beta^1} \mathbb{R}$ (defined by the same formula).

10.2. **The 3-cyclic cocycle.** Let η be a G-invariant probability on $X^G_{\mathcal{P}(\omega)}$. Define

$$\tau_{w,\eta} : \mathcal{A}^G_{\mathcal{P}(\omega)} \times \mathcal{A}^G_{\mathcal{P}(\omega)} \to \mathbb{C}; (f, g) \mapsto \int Y(f) g d\eta.$$

Using the Leibnitz rules and the invariance of G, it is straightforward to check that $\tau_{w,\eta}$ is 1-cyclic cocycle. In [6] was constructed for a smooth action α on a Frechet algebra $\mathcal{A}$ a homomorphism $H^n_\lambda(\mathcal{A}) \to H^{n+1}_\lambda(\mathcal{A} \rtimes_\alpha \mathbb{R})$, where $H^*_\lambda(\bullet)$ stands for the cyclic cohomology. This homomorphism is indeed induced by a homomorphism at the level of cyclic cocycles $\#_\alpha : Z^n_\lambda(\mathcal{A}) \to Z^{n+1}_\lambda(\mathcal{A} \rtimes_\alpha \mathbb{R})$ and commutes with the periodisation operator S. Hence it gives rise to a homomorphism in periodic cohomology $HP^*(\mathcal{A}) \to HP^{*+1}(\mathcal{A} \rtimes_\alpha \mathbb{R})$ which turns out to be an isomorphism. This isomorphism is for periodic cohomology the analogue of the Thom-Connes isomorphism in K-theory.

We give now the description of $\#_{\beta^0} \tau_{w,\eta}$. Let us define first

$$X_{\beta^0} : \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \to \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$$

and

$$Y_{\beta^0} : \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \to \mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$$

respectively by $X_{\beta^0} f(t) = X(f)(t)$ and $Y_{\beta^0} f(t) = Y(f)(t)$, for all f in $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$ and t in $\mathbb{R}$. Using the relation $Y \circ \beta^0_t = \beta^0_t \circ Y - t \ln 2 \, \beta^0_t \circ X$ and applying the definition of $\#^0_\beta$ (see [6, section 3.3]), we get:

Proposition 10.2. *For any elements f, g and h in $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \rtimes \mathbb{R}$, we have*

(1)

$$\#_{\beta^0} \tau_{w,\eta}(f, g, h) = -2\pi i \eta(Y_{\beta^0} f * g * Z_{\beta^0} h(0) + Z_{\beta^0} f * g * Y_{\beta^0} h(0)$$
$$-2\pi i \ln 2 (\eta(1/2 {Z_{\beta^0}}^2 f * g * X_{\beta^0} h(0) + Z_{\beta^0} f * Z_{\beta^0} g * X_{\beta^0}(h)(0) - 1/2 X_{\beta^0} f * g * {Z_{\beta^0}}^2(0))$$

(2)
$$\#_{\beta^0}\tau_{w,\eta}(\beta^1_t f, \beta^1_t g, \beta^1_t h) = \#_{\beta^0}\tau_{w,\eta}(f,g,h)$$
for all t in $\mathbb{R}$, i.e the cocycle $\#_{\beta^0}\tau_{w,\eta}$ is β^1-invariant.

According to [6, Section 7.2], the action β^1 on $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \rtimes \mathbb{R}$ also gives rise to a homomorphism $\#_{\beta^1} : Z^n_\lambda(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}) \to Z^{n+1}_\lambda(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R})$ which induces an isomorphim $HP^*(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}) \xrightarrow{\cong} HP^{*+1}(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R})$. A direct application of the definition of $\#_{\beta^1}$ leads to

Lemma 10.3. *Let ϕ be a β_1-invariant 3-cyclic cocycle for $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}$. Let us define for any f, g and h in $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}$.*
$$\widetilde{\phi}(f,g,h) = 2\pi\imath \int_{t_0+t_1+t_2=0} f(t_0)\beta^1_{t_0} g(t_1)\beta^1_{-t_2}(t_2).$$
Then
$$\begin{aligned}\#_{\beta^1}\phi(f_0,f_1,f_2,f_3) = &-\widetilde{\phi}(f_0,f_1,f_2 * Z_{\beta^1} f_3) + \widetilde{\phi}(Z_{\beta^1} f_0 * f_1, f_2, f_3) \\ &- \widetilde{\phi}(f_0, Z_{\beta^1} f_1 * f_2, f_3) - \widetilde{\phi}(Z_{\beta^1} f_0, f_1 * f_2, f_3)\end{aligned}$$

Definition 10.4. *With above notations, the 3-cyclic cocycle on $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}$ associated to the Penrose hyperbolic tiling coloured by w and to a G-invariant probability η on $X^G_{\mathcal{P}(\omega)}$ is*
$$\phi_{w,\eta} = \#_{\beta^1}\#_{\beta^0}\tau_{w,\eta}.$$

Notice that if we carry out this construction for a tiling $\mathcal{T}$ of the Euclidian space with continuous hull $X^{\mathbb{R}^2}_{\mathcal{T}}$ with respect to the $\mathbb{R}^2$-action by translations, we get taking twice the crossed product by $\mathbb{R}$ a 3-cyclic cocycle which is indeed equivalent (via the periodisation operator) to the 1-cycle cocycle on $C(X^{\mathbb{R}^2}_{\mathcal{T}}) \rtimes \mathbb{R}^2 \cong (C(X^{\mathbb{R}^2}_{\mathcal{T}}) \rtimes \mathbb{R}) \rtimes \mathbb{R}$ arising from the trace on $C(X^{\mathbb{R}^2}_{\mathcal{T}}) \rtimes \mathbb{R}$ associated to an $\mathbb{R}$-invariant probability on $X^{\mathbb{R}^2}_{\mathcal{T}}$.

The class $[\phi_{w,\eta}]$ of $\phi_{w,\eta}$ in $HP^1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R})$ is the image of the class of $\tau_{w,\eta}$ under the composition of isomorphism
$$HP^1(\mathcal{A}^G_{\mathcal{P}(\omega)}) \xrightarrow{\cong} HP^0(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R}) \xrightarrow{\cong} HP^1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}).$$
Since pairing with periodic cohomology provides linear forms for K-theory groups, the 3-cyclic cocycle $\phi_{w,\eta}$ provides a linear map
$$\phi_{w,\eta,*} : K^1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}) \to \mathbb{C};\ x \mapsto \langle [\phi_{w,\eta}], x\rangle.$$
The main issue in computing $\phi_{w,\eta,*}(K^1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}))$ is that the Thom-Connes isomorphism a priori may not hold for $K^1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R})$. If it were the case, the inclusion $\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R} \hookrightarrow C(X^G_{\mathcal{P}(\omega)}) \rtimes G$ would induces an isomorphism $K_1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R}) \xrightarrow{\cong} K_1(C(X^G_{\mathcal{P}(\omega)}) \rtimes G)$ and from this we could get that
$$\phi_{w,\eta,*}(K_1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^\rho_{\beta^1} \mathbb{R})) = \mathbb{Z}[\hat{\eta}] \overset{\text{def}}{=\!=} \{\hat{\eta}(E),\ E \text{ compact-open subset of } Z_w\},$$
where $\hat{\eta}$ is the probability on Z_w of proposition 4.4 in one-to-one correspondance with η.

Since $\mathbb{Z}[\hat{\eta}]$ is indeed the one dimension gap-labelling for the subshift corresponding to w, this would be viewed as an odd version of the gap labelling. Nevertheless, the right setting to state this generalisation of the gap-labelling seems to be the Frechet algebra and a natural question is whether we have

$$\{\langle[\phi_{w,\eta}], x\rangle;\, x \in K_1(\mathcal{A}^G_{\mathcal{P}(\omega)} \rtimes_{\beta^0} \mathbb{R} \rtimes^{\rho}_{\beta^1} \mathbb{R})\} = \mathbb{Z}[\hat{\eta}]$$

or if the pairing bring in new invariants.

Aknowledgments

This paper was partially written during a visit of the first author at the Pacific Institut for Mathematical Sciences in Victoria, Canada. He is also indebted to J. Renault for some very helpful discussions on Haar systems for groupoids.

References

[1] J.E. Anderson, I.F. Putnam. *Topological invariants for substitution tilings and their associated C^*-algebras*, Ergod. Th. & Dyn. Syst. **18** (1998), 509-537.
[2] J. Bellissard. *Gap labelling theorems for Schrödinger operators, From number theory to physics* (Les Houches, 1989), 538?630, Springer, Berlin, 1992.
[3] J. Bellissard, R. Benedetti, J.-M. Gambaudo. *Spaces of tilings, finite telescopic approximations and gap-labelling,* Comm. Math. Phys. **261** (2006), no. 1, 1–41.
[4] M. T. Benameur, H. Oyono-Oyono. *Index theory for quasi-crystals I. Computation of the gap-label group*, Journal of Functional Analysis 252 (2007), 137-170.
[5] R. Benedetti, J. M. Gambaudo. *On the dynamics of $\mathbb{G}$-solenoids. Applications to Delone sets*, Ergod. Th. & Dyn. Syst. **23** (2003), 673-691.
[6] G. A. Elliott, T. Natsume, R Nest. *Cyclic cohomology for one-parameter smooth crossed products* Acta Math. **160**, (1988), n° 3-4, 285-305
[7] L. Garnett. *Foliations, The Ergodic theorem and brownian motion*, Journ. of Funct. Analysis. **51** (1983), 285-311
[8] W.H. Gottschalk, G.A. Hedlund. *Topological dynamics.* American Mathematical Society Colloquium Publications, Vol. 36. American Mathematical Society, Providence, R. I., 1955
[9] É. Ghys. *lamination par surface de Riemann*, Dynamique et géométrie complexes, Panoramas & Synthèse **8** (1999), 49-95
[10] J. Kaminker, I. Putnam. *A proof of the gap labelling conjecture*, Michigan Mathematical Journal **51**, (2003), no 3, 537–546.
[11] G. G., Kasparov. *Equivariant KK-theory and the Novikov conjecture.* Invent. Math. **91** (1988), no. 1, 147–201.
[12] J. Kellendonk, I.F. Putnam. *Tilings, C^*-algebras and K-theory*, Directions in Mathematical Quasicrystals, CRM Monograph Series **13** (2000), 177-206, M.P. Baake & R.V. Moody Eds., AMS Providence.
[13] G. Margulis, S. Mozes. *Aperiodic tiling of the hyperbolic plane by convex polygons*, Israel Journ. of Math. **107** (1998), 319-325
[14] S. Petite. *On invariant measures of finite affine type tilings*, Ergod. Th. & Dyn. Syst. (2006) **26**, 1159-1176
[15] R. Penrose. *Pentaplexity*, Mathematical Intelligencer **2** (1979), 32-37
[16] J. F. Plante *Foliations with measure preserving holonomy*, Ann. of Math. (2)**102** (1975), 327–361
[17] I. Putnam. *The C*-algebras associated with minimal homeomorphisms of the Cantor set*, Pacific J. Math. **136** (1989), 329-352.
[18] M. Pimsner, D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras*, J. Operator Theory 4 (1980), n°. 1, p. 93–118.
[19] M. Queffélec. *Substitution dynamical systems — spectral analysis*, LNM, 1294. Springer-verlag, Berlin, 1987
[20] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, 793, Springer, Berlin.

[21] E. A. Robinson, Jr. *Symbolic dynamics and tilings of* $\mathbb{R}^d$, Symbolic dynamics and its applications, Proc. Sympos. Appl. Math., **60**, Amer. Math. Soc., Providence , RI, 2004, p. 81–119.
[22] D. Sullivan. *Cycles for the dynamical study of foliated manifolds and complex manifolds*, Invent. Math. **36** (1976), 225-255.
[23] A. Valette. *Introduction to the Baum-Connes conjecture*, Lectures in Mathematics ETH Zürich, From notes taken by Indira Chatterji, with an appendix by Guido Mislin, Birkhäuser Verlag, Basel, 2002.

Laboratoire de Mathématiques, Université Blaise Pascal & CNRS (UMR 6620), Les Cézeaux, 24 avenue des Landais, BP 80026, 63177 Aubière Cedex, France, phone:+33.4.73.40.70.71, fax:+33.4.73.40.70.64
E-mail address: oyono@math.cnrs.fr

L.A.M.F.A, Université de Picardie Jules Verne & CNRS, Amiens, France, 33 rue Saint-Leu, 80039 Amiens Cedex 1, France, phone:+33.3.22.82.78.15 fax:+33.3.22.82.78.38
E-mail address: samuel.petite@u-picardie.fr